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ON | C, α, γ $|_k$ SUMMABILITY FACTORS OF FOURIER SERIES

By

Ashok Ganguly

Department of Mathematics,
G. S. Institute of Technology and Science,
Indore-452 003, M. P. , India

(*Received : May 28, 1985 ; Revised : July 3, 1986*)

1. INTRODUCTION

Let Σu_n be a given series with the sequence of partial sums $\{S_n\}$ and let $t_n = t_n^0 = nu_n$. By $\{ S_n^\alpha \}$ and $\{ t_n^\alpha \}$ we denote the n-th Cesāro means of order α ($\alpha > -1$) of the sequences $\{ S_n\}$ and $\{t_n\}$ respeetively . Thus

$$S_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} s_\nu ,$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} .$$

The series Σu_n is said to be summable $| C, \alpha, \gamma |_k$, $k \geqslant 1$, $\alpha > -1$ and $\gamma \geqslant 0$, if

$$(1.1) \quad \sum_n n^{k\gamma + k - 1} \left| S_n^\alpha - S_{n-1}^\alpha \right|^k < \infty \quad \text{(Flett [5])} .$$

By virtue of the identity

$$t_n^{\alpha} = n\left(S_n^{\alpha} - S_{n-1}^{\alpha}\right) \quad \text{(Kogbetliantz [7])}$$

(1.1) can also be written as

(1.2) $\sum_n n^{k\gamma-1} \mid t_n^{\alpha} \mid^k < \infty$.

For $\gamma = 0$, the summability $\mid C, \alpha, \gamma \mid_k$ reduces to the summability $\mid C, \alpha \mid_k$ (see [6]) ; for $\gamma = 0$ and $k = 1$, it reduces to the summability $\mid C, \alpha \mid$.

2. THE MAIN RESULT

Let the Fourier series associated with $f(t)$ be

$$1/2\, a_0 + \sum_{n=1}^{\infty} (a_n \operatorname{Cos} nt + b_n \operatorname{Sin} nt) = \sum_{n=0}^{\infty} A_n(t) .$$

We write

$$\phi(t) = 1/2\, \{ f(x+t) + f(x-t) - 2S \} .$$

A sequence $\{ \mu_n \}$ is said to be convex, if $\Delta^2 \mu_n \geqslant 0$; $n = 1, 2, \ldots$, where $\Delta \mu_n = \mu_n - \mu_{n+1}$ and $\Delta^2 \mu_n = \Delta(\Delta \mu_n)$.

The objet of this paper is to prove the following :

Theorem . *If* $\{ \mu_n \}$ *is a convex sequence such that*

$\sum \frac{\mu_n}{n} < \infty$, *then the series* $\sum \frac{\mu_n A_n(t)}{n^{1-\alpha+\gamma} (\operatorname{Log} n)^{\beta/k}}$ *at* $t = x$

is summable $| C, \alpha, \gamma |_k$, *where* $0 \leqslant \gamma < \alpha < 1$ *and* $k \geqslant 1$, *provided that*

$$(2.1) \quad \int_0^t | \phi (u) |^k \, du = O \left\{ t \left(\log \frac{1}{t} \right)^{\beta} \right\}, \ \beta \geqslant 0 , \ t \to 0 .$$

For $k = 1$, $\gamma = 0$, our theorem reduces to the result of Dikshit [4] which is an extension of a theorem of Cheng [2] .

3. PREREQUISITES

We require the following lemmas :

Lemma 1 [3] . *If* $0 < \alpha < 1$, $0 < t < 2\pi$, *and*

$$R_n^{\alpha} (t) = \sum_{r=1}^{n} A_{n-r}^{\alpha-1} \, r \cos rt \ ,$$

then

$$R_n^{\alpha} (t) = \begin{cases} O \, (n^2) \text{ for all } t > 0 , \\ O \, (n t^{-\alpha}) \text{ for } t > 1/n . \end{cases}$$

Lemma 2 [1] . *If* $0 \leqslant \alpha \leqslant 1$ *and* $0 \leqslant m \leqslant n$, *then*

$$\left| \sum_{\nu=0}^{m} A_{n-\nu}^{\alpha-1} a_{\nu} \right| \leqslant \max_{0 \leqslant r \leqslant m} \left| \sum_{\nu=0}^{r} A_{r-\nu}^{\alpha-1} a_{\nu} \right| .$$

Lemma 3 . *Let* $0 < \alpha < 1$ *and* $0 < t \leqslant 2\pi$, *and write*

$$M_a^{\alpha} (t) = \frac{1}{A_n^{\alpha}} \sum_{\nu=2}^{n} A_{n-\nu}^{\alpha-1} \frac{\mu_{\nu}}{(\log \nu)^{\beta/k}} \nu^{\alpha-\gamma} \operatorname{Cos} \nu t , \ \beta \geqslant 0 ,$$

$0 \leqslant \gamma < \alpha < 1$ *and* $k \geqslant 1$. *Then*

$$M_n(t) = \begin{cases} (3.1) \quad O\left\{ n^{-\alpha} \sum_{\nu=2}^{n} \dfrac{\nu^{1+\alpha-\gamma} \Delta \mu_\nu}{(\log \nu)^{\beta/k}} \right\} + O\left\{ \dfrac{n^{1-\gamma}\ \mu_n}{(\log n)^{\beta/k}} \right\} & \text{for } 0 < t \leqslant 1/n, \\ (3.2) \quad O\left\{ (nt)^{-\alpha} \sum_{\nu=2}^{n} \dfrac{\nu^{\alpha-\gamma}}{(\log \nu)^{\beta/k}} \Delta \mu_\nu \right\} + O\left\{ -\dfrac{\mu_n}{n^{\gamma}\ t^{\alpha}\ (\log n)^{\beta/k}} \right\} & \text{for } t > 1/n . \end{cases}$$

By applying Abel's transformation and taking the help of lemmas 1 and 2, Lemma 3 can easily be proved.

Lemma 4. *If the condition (2.1) holds, then (for $k \geqslant 1$)*

$$\text{(i)} \qquad \left\{ \int_0^{1/n} |\phi(t)|\, dt \right\}^k = O\left\{ \frac{(\log n)^\beta}{n^k} \right\} \quad ;$$

and, for $k \geqslant 1$ and $0 < \alpha < 1$,

$$\text{(ii)} \qquad \left\{ \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^{\alpha}} dt \right\}^k = O\left\{ (\log n)^\beta \right\} \quad .$$

These results follow by an integration by parts or by an application of Hölder's inequality, as necessary.

Lemma 5. *If $\{\mu_n\}$ is a convex sequence such that $\Sigma \frac{\mu_n}{n} < \infty$, then*

$$\text{(i)} \qquad \sum_{n=1}^{m} \log(n+1)\, \Delta \mu_n = O(1) \text{ as } m \to \infty, \text{ (See Pati [8]) } ;$$

further, the hypotheses on $\{\mu_n\}$ imply

$$\text{(ii)} \qquad \sum_{n=1}^{m} \Delta \mu_n = O(1),\ m \to \infty ,$$

(iii) $\sum\limits_{n=1}^{m} \frac{(\mu_n)^k}{n} = O(1), \; m \to \infty .$

The proof of (ii) and (iii) follow easily .

4. PROOF OF THE THEOREM

We denote the n-th Cesàro mean of order α of the sequence $\{ n^{c-\gamma} \mu_n A_n (x) (\log n)^{-\beta/k} \}$ by $C_n^\alpha (x)$. Then we have to show that

$$\Sigma \, n^{k\gamma-1} \, | \, C_n^\alpha (x) \, |^k < \infty \quad \text{for } k \geqslant 1 .$$

We have

$$C_n^\alpha (x) = \frac{2}{\pi} \int_0^\pi \phi(t) \, \frac{1}{A^\alpha} \sum_{\nu=2}^{n} \frac{A_{n-\nu}^{\alpha-1} \, \nu^{\alpha-\gamma} \, \mu_\nu \cos t}{(\log \nu)^{\beta/k}} \, dt$$

$$= \frac{2}{\pi} \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \phi(t) \, M_n^\alpha (t) \, dt$$

$$= Q_n^1 + Q_n^2 , \quad say .$$

In view of Minkowski's inequality , it is sufficient to prove that

(4.1) $\Sigma \, n^{k\gamma-1} \, | \, Q_n^1 \, |^k < \infty$, and

(4.2) $\Sigma \, n^{k\gamma-1} \, | \, Q_n^2 \, |^k < \infty$.

Proof of (4.1) . Using (3.1) we have

$$\sum_{n=2}^{m} n^{k\gamma-1}\left|\,Q_n^1\right|^k \leqslant \sum_{n=2}^{m} n^{k\gamma-1}\left[\frac{2}{\pi}\int_0^{1/n}\left|\,\phi(t)\right| dt\right.$$

$$\left.\left\{O\left(n^{-\alpha}\sum_{\nu=2}^{n}\frac{\nu^{1+\alpha-\gamma}\,\Delta\mu_\nu}{(\log\nu)^{\beta/k}}\right)+O\left(\frac{n^{1-\gamma}\,\mu_n}{(\log n)^{\beta/k}}\right)\right\}\right]^k$$

$$=\left\{O\left[\sum_{n=2}^{m} n^{k\gamma-1}\left(\int_0^{1/n}\frac{|\,\phi(t)\,|}{n^{\alpha}}\sum_{\nu=2}^{n}\frac{\nu^{1+\alpha-\gamma}\,\Delta\mu_\nu}{(\log\nu)^{\beta/k}}\,dt\right)^k\right]^{1/k}\right.$$

$$\left.+\,O\left[\sum_{n=2}^{m} n^{k\gamma-1}\left(\int_0^{1/n}\frac{|\,\phi(t)\,|\,n^{1-\gamma}\,\mu_n}{(\log n)^{\beta/k}}\,dt\right)^k\right]^{1/k}\right\}^k$$

$$=\left\{M_1^{1/k}+M_1^{1/k}\right\}^k,\quad say\ .$$

Now, applying Lemmas 4 (i) and 5 (i), we get

$$M_1=O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k(\alpha-\gamma)}}\left(\int_0^{1/n}|\,\phi(t)\,|\,dt\right)^k\left(\sum_{\nu=2}^{n}\frac{\nu^{1+\alpha-\gamma}\,\Delta\mu_\nu}{(\log\nu)^{\beta/k}}\right)^k\right]$$

$$=O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k(\alpha-\gamma)}}\cdot\frac{(\log n)^{\beta}}{n^k}\left(\sum_{\nu=2}^{n}\frac{\nu^{k(1+\alpha-\gamma)}\,\Delta\mu_\nu}{(\log\nu)^{\beta}}\right)\left(\sum_{\nu=2}^{n}\Delta\mu_\nu\right)^{k-1}\right]$$

$$=O\left[\sum_{\nu=2}^{m}\frac{\nu^{k(1+\alpha-\gamma)}\,\Delta\mu_\nu}{(\log\nu)^{\beta}}\cdot\sum_{n=\nu}^{m}\frac{(\log n)^{\beta}}{n^{1+k(1+\alpha-\gamma)}}\right]$$

$$=O\left[\sum_{\nu=2}^{m}\Delta\mu_\nu\right]$$

$$=O\ (1)\ as\ m\to\infty\text{, by Lemma 5 (ii) .}$$

Next , using Lemma 4 (i) again, we get

$$M_2 = O\left[\sum_{n=2}^{m} \frac{n^{k-1}\,\mu_n^k}{(\log n)^{\beta}}\left(\int_0^{1/n} |\,\phi(t)\,|\,dt\right)^k\right]$$

$$= O\left[\sum_{n=2}^{m} \frac{\mu_n^k}{n}\right]$$

$= O(1)$ as $m \to \infty$, by Lemma 5 (iii) .

Proof of (4.2) : Applying (3.2), we obtain

$$\sum_{n=2}^{m} n^{k\gamma-1}\,|\,Q_n^2\,|^k \leqslant \sum_{n=2}^{m} n^{k\gamma-1}\left[\frac{2}{\pi}\int_{1/n}^{\pi} |\,\phi(t)\,|\,dt\right.$$

$$\left.\left\{O\left((nt)^{-\alpha}\sum_{\nu=2}^{n}\frac{\nu^{\alpha-\gamma}\,\Delta\mu_\nu}{(\log \nu)^{\beta/k}}\right) + O\left(\frac{\mu_n}{t^{\alpha}\,n^{\gamma}\,(\log n)^{\beta/k}}\right)\right\}\right]^k$$

$$= \left\{O\left[\sum_{n=2}^{m} n^{k\gamma-1}\left(\frac{1}{n^{\alpha}}\int_{1/n}^{\pi}\frac{|\,\phi(t)\,|}{t^{\alpha}}\sum_{\nu=2}^{n}\frac{\nu^{\alpha-\gamma}\,\Delta\mu_\nu}{(\log \nu)^{\beta/k}}\,dt\right)^{k}\right]^{1/k}\right.$$

$$\left. + O\left[\sum_{n=2}^{m} n^{k\gamma-1}\left(\frac{1}{n^{\gamma}}\int_{1/n}^{\pi}\frac{|\,\phi(t)\,|}{t^{\alpha}}\,\frac{\mu_n}{(\log n)^{\beta/k}}\,dt\right)^k\right]^{1/k}\right\}^k$$

$$= \left\{N_1^{1/k} + N_2^{1/k}\right\}, \text{ say} .$$

Using Lemmas 4 (ii), and 5 (i), we get

$$N_1 = O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k(\alpha-\gamma)}}\left(\int_{1/n}^{\pi}\frac{|\,\phi(t)\,|}{t^{\alpha}}\,dt\right)^k\left(\sum_{\nu=2}^{n}\frac{\nu^{\alpha-\gamma}\,\Delta\mu_\nu}{(\log \nu)^{\beta/k}}\right)^k\right]$$

$$= O\left[\sum_{n=2}^{m}\frac{(\log n)^{\beta}}{n^{1+k(\alpha-\gamma)}}\left(\sum_{\nu=2}^{n}\frac{\nu^{k(\alpha-\gamma)}\,\Delta\mu_\nu}{(\log \nu)^{\beta}}\right)\left(\sum_{\nu=2}^{n}\Delta\mu_\nu\right)^{k-1}\right]$$

$$= O\left[\sum_{\nu=2}^{m} \frac{\nu^{k(\alpha-\gamma)}\,\Delta\mu_\nu}{(\log \nu)^{\beta}} \sum_{n=\nu}^{m} \frac{(\log n)^{\beta}}{n^{1+k(\alpha-\gamma)}}\right]$$

$$= O\left[\sum_{\nu=2}^{m} \Delta\mu_\nu\right]$$

$= O(1)$, as $m \to \infty$, by Lemma 5 (ii) .

Further, applying Lemma 4 (ii) again, we have

$$N_2 = O\left[\sum_{n=2}^{m} \frac{\mu_n^k}{n} \frac{1}{(\log n)^{\beta}} \left(\int_{1/n}^{\pi} \frac{|\phi(t)|}{t^{\alpha}}\, dt\right)^{k}\right]$$

$$= O\left[\sum_{n=2}^{m} \frac{\mu_n^k}{n}\right]$$

$= O(1)$, as $m \to \infty$, by Lemma 5 (iii) .

Thus the proof of the theorem is complete .

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SOME THEOREMS ON BILATERAL GENERATING FUNCTIONS

By

B. D. Agrawal and Sanchita Mukherjee

Department of Mathematics, Faculty of Science,
Banaras Hindu University, Varanasi - 221005, India

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ABSTRACT

In an earlier paper [1], we defined a general polynomial system $S_n^A\,[\,(x_m), y\,]$ with the help of a generating relation involving the Lauricella function F_A. In the present paper, we establish an effective method of obtaining bilateral generating relations in terms of Pochhammer's double-loop integral .

1. INTRODUCTION

The general polynomial system $\{\, S_n^A\,[\,(x_m), y\,)/n=0, 1, 2 \,..\,\}$ is defined by means of generating reletion [1] :

$$\sum_{n=0}^{\infty} S^{A\,;\,m,\,a,\,(b_m),\,(\mu_m)}_{n,\,\nu,\,(\gamma_m\,;\,1),\,(c_m)}\,[\,(x_m), y\,]\, t^n$$

$$= e^{\nu y t}\, F_A\,[\,a, (b_m)\,; (x_m)\,; \mu_1 x_1 t\,,\ \mu_2 x_2 t^{r_2}, \ldots, \ \mu_m x_m t^{r_m}\,]\,, \quad (1.1)$$

where F_A is one of the Lauricella functions [3, p.41] . The conditions for validity of (1.1) are

(i) (x_m), y, t are all finite quantities.

(ii) m, $(r_{m;1})$ are natural numbers ;

(iii) ν, (μ_m) are finite numbers ;

(iv) a, (b_m) are non-zero numbers ;

(v) (c_m) are neither zero nor negative integers

For this polynomial system, we have an explicit representation in the form [1] :

$$S_n^A\,[\,(x_m),\,y\,] = \sum_{m_1=0}^{n}\ \sum_{m_2=0}^{[(n-m_1)/r_2]} \cdots \sum_{m_m=0}^{[(n-m_1-\sum_2^{m-1} r_i m_i)/r_m]} \frac{(a)_{n-m_1-\sum_2^m (r_i-1)\,m_i}\ (b_.)_{n-m_1-\sum_2^m r_i m_i}}{(n-m_1-\sum_2^m r_i m_i)\,!\ (c_1)_{n-m_1-\sum_2^m r_i m_i}}$$

$$\frac{[\,(b_{m;1})_{(m_{m;1})}\,]\,(\nu y)^{m_1}\ (\mu_1 x_1)^{n-m_1-\sum_2^m r_i m_i}\ [\,(\mu_{m;1}\,x_{m;1})^{m_{m;1}}\,]}{[\,(c_{m;1})_{(m_{m;1})}\,]\ [\,(m_m\,!)\,]} \tag{1.2}$$

Saran [5], Shankar [6], Panda [4], Srivastava ([8], [9]) and Singhal ([7], [12]) and Srivastava and Lavoie [10] have established a number of theorems on bilateral as well as bilinear generating-functions (see also Chapter 8 of the latest work on the subject by Srivastava and Manocha [11]) . Recently, Baweza and Saran [2] gave some interesting theorems for obtaining bilateral generating functions in terms of Pochhammer's double-loop integrals (see, for details, [13,p. 256]) . The object of this paper is to derive analogous generating functions for the polynomial systems defined by (1.1) .

We employ the following notations in our present investigation :

(i) $(m) = 1, 2, \ldots, m$;

(ii) $(a_p) = a_1, a_2, \ldots, a_p$.

(iii) $[(a_p)]_n = (a_1)_n (a_2)_n \ldots (a_p)_n$.

(iv) $$[(a_{m;i})_{(p_{m;i})}] = (a_1)_{p_1} (a_2)_{p_2} \ldots (a_{i-1})_{p_{i-1}} (a_{i+1})_{(p_{i+1})} \ldots (a_m)_{p_m}$$

(v) $$[(a_m x_m)^{p_m}] = (a_1 x_1)^{p_1} (a_2 x_2)^{p_2} \ldots (a_m x_m)^{p_m}$$

(vi) $$[(a_{m;i}\, x_{m;i})^{p_{m;i}}] = (a_1 x_1)^{p_1} (a_2 x_2)^{p_2} \ldots (a_{i-1}\, x_{i-1})^{p_{i-1}} (a_{i+1}\, x_{i+1})^{p_{i+1}} \ldots (a_m x_m)^{p_m}$$

2. THE MAIN RESULTS

We shall now derive four theorems on bilateral generating functions .

THEOREM 1, *If*

$$F [x_m), y;t] = \sum_{n=0}^{\infty} S_n^A [(x_m), y] t^n, \tag{2.1}$$

then

$$\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta)}{(2\pi i)^2}\int_0 (-p)^{\alpha-1}\,(p-1)^{\beta-1}$$

$${}_2F_1\left[\begin{matrix}1-\beta,\ \gamma; & \dfrac{zp}{(p-1)\,(z-1)}\\ \alpha; & \end{matrix}\right] F\,[\,(x_m),\ y;\ t_p\,]\ dp$$

$$= (1-z)^{\gamma}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\alpha+\beta)_n}\ {}_2F_1\left[\begin{matrix}-n,\ \gamma; & z\\ \alpha; & \end{matrix}\right] S_n^A\,[\,(x_m),\ y\,]\ t^n, \quad (2.2).$$

where the integration is aken over Pochhammer's double-loop C.

Proof . Let us write tp for t in (2.1), multiply it by a constant multiple of $\int_0 (-p)^{\alpha-1}\,(p-1)^{\beta-1}\ {}_2F_1\left[\begin{matrix}1-\beta,\ \gamma; & \dfrac{zp}{(z-1)\,(p-1)}\\ \alpha; & \end{matrix}\right] dp$,

integrate with respect to p along the Pochhammer's double-loop C. Using the known result [13, p. 256]

$$\left(\frac{1}{2\pi i}\right)^2\int_C (-p)^{\alpha-1}\,(p-1)^{\beta-1}\ dp = \frac{1}{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta)} \qquad (2.3)$$

and simplifying, we get the desired result (2.2), on employing a known transformation for ${}_2F_1$. Alternatively, we may put (2.2) in the form :

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\alpha+\beta)_n}\ {}_2F_1\left[\begin{matrix}-n;\gamma; & z\\ \alpha; & \end{matrix}\right] S_n^A\,[\,(x_m),\ y\,)\ t^n$$

$$= (1-z)^{-\gamma}\ F^{\,2:1;0;1\ \ldots;1}_{\,1:1;0;1;\ \ldots;1}\left[\begin{matrix}(\alpha:1,1,r_2,\ldots,r_m),\\ (\alpha+\beta:0,1,r_2,\ldots,r_m),\end{matrix}\right.$$

$$(a:0.\,0,1,\ldots,1):(\gamma:1);-;)\,b_1:1);\ldots;(b_n:1);$$

$$(\alpha:1);-;(c_1:1);\ldots;(c_m:1);$$

$$\frac{z}{z-1}\ , \nu yt\ , \mu_1 x_1 t,\ \mu_2 x_2 t^{r_2}\ , \ldots\ , \mu_m x_m t^{r_m} \Bigg] , \qquad (2.5)$$

in terms of a generalized Lauricella function of Srivastava and Daoust (*Cf.*, *e. g.*, [3, p. 106]) .

THEOREM 2, *If*

$$G\,[\,(x_m)\,,\, y\,;\,]\; t = \sum_{n=0}^{\infty} S_n^A\;[\,(x_m)\,,\, y\,]\; t^n\,, \qquad (2.6)$$

then

$$\frac{\Gamma(1-\alpha_1)\,\Gamma(1-\beta_1)\,\Gamma(1-\alpha_2)\,\Gamma(1-\beta_2)\,\Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_2+\beta_2)}{(2\pi i)^4}$$

$$\int_C \int_{C'} (-p)^{\alpha_1-1}\,.\,(p-1)^{\beta_1-1}\,(-q)^{\alpha_2-1}\,(q-1)^{\beta_2-1}$$

$$ {}_2F_1\left[\begin{matrix} 1-\beta_1,\ 1-\beta_2\ ; \\ \alpha\ ; \end{matrix}\ \frac{pq\,z}{(z-1)\,(p-1)\,(q-1)}\right]$$

$$.\ G\,[\,(x_m)\,,\, y\,,\ \frac{pqt}{1-z}\,]\ dp\,dq$$

$$= (1-z)^{\alpha_2} \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\,(\alpha_2)_n}{(\alpha_1+\beta_1)_n\,(\alpha_2+\beta_2)_n}$$

$$ {}_2F_1\left[\begin{matrix} -n,\ \alpha_2+n\ ; \\ \alpha_1\ ; \end{matrix}\ z\right] S_n^A\ [\,(x_m)\,,\, y\,]\ t^n\,, \qquad (2.7)$$

where c and c′ denote Pochhamner's double-loops .

The proof of Theorem 2 is very similar to that of Theorem 1. Theorem 2 also yields a new bilateral generating relation :

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n}{(\alpha_1+\beta_1)_n (\alpha_2+\beta_2)_n} \, {}_2F_1\left[\begin{matrix} -n, \alpha_2+n ; \\ \alpha_1 ; \end{matrix} \; z\right] S_n^A [(x_m), y] \, t^n$$

$$= (1-z)^{-\alpha_2} \, F^{3:0;1;..;1}_{2:1;\;,\ldots,;1}\left[\begin{matrix} (\alpha_1 : 1, 1, r_2, .., r_m), \\ (\alpha_1+\beta_1 : 0, 0, r_2, .., r_m), \end{matrix}\right.$$

$$(\alpha_2 ; 1, 1, r_2, .., r_m), (a : 0, 0, 1, .., 1) :-; (b_1 : 1) ; .. ; (b_m : 1) ;$$

$$(\beta_2+\alpha_2 : 0, 0, r_2, .., r_m), - :-; (c_1 : 1) ; \ldots ; (c_m : 1) ;$$

$$\left. \frac{z}{z-1}, \; \frac{\nu y t}{1-z}, \; \frac{\mu_1 x_1 t}{1-z}, \; \frac{\mu_2 x_2 \, t^{r_2}}{(1-z)^{r_2}}, \ldots, \frac{\mu_m x_m \, t^{r_m}}{(1-z)^{r_m}} \right], \qquad (2.8)$$

in terms of the aforementioned generalized Lauricella function.

Similar are the proofs of the following theorems :

THEOREM 3. *If*

$$F[(x_m), y ; t] = \sum_{n=0}^{\infty} S_n^A [(x_m), y] \, t^n, \qquad (2.9)$$

then

$$\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta)}{(2\pi i)^2} \int_C (-p)^{\alpha-1} (p-1)^{\beta-1}$$

$$. F_D \left[1-\beta, a_1, \ldots, \alpha_n ; \gamma ; \frac{z_1 p}{(z_1-1)(p-1)}, \ldots, \right.$$

$$\left. \frac{z_n p}{(z_n-1)(p-1)} \right] F_1 [(x_m), y ; t] \, dp$$

$$= (1-z_1)^{\alpha_1} \ldots (1-z_n)^{\alpha_n} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\alpha+\beta)_n}$$

$$F_D [-n, a_1, \ldots, \alpha_n ; \gamma ; z_1, \ldots, z_n] \, . \, S_n^A [(x_m), y] \, t^n \qquad (2.10)$$

THEOREM 4. *If*

$$H [(x_m), y ; t] = \sum_{n=0}^{\infty} S_n^A [(x_m), y] \, t^n , \qquad (2.11)$$

then

$$\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta) \prod_{i=1}^{n} \left\{ \Gamma(1-\alpha_1)\,\Gamma(1-\beta_i)\,\Gamma(\alpha_i+\beta_i) \right\}}{(2\pi i)^{2n+2}}$$

$$\int_{C_1} \cdots \int_{C_{n+1}} (-p)^{\alpha-1} (p-1)^{\beta-1} \left\{ \prod_{i=1}^{n} (-p_i)^{\alpha_i-1} (p_i-1)^{\beta_i-1} \right\}$$

$$. F_A \left[1-\beta, 1-\beta_1, \ldots, 1-\beta_n ; a_1, \ldots, \alpha_n ; \right.$$

$$\frac{pp_1 \; z_1}{(z_1+\ldots+z_n-1)(p-1)(p_1-1)}, \ldots$$

$$\ldots, \frac{pp_n \; z_n}{(z_1+\ldots+z_n-1)(p-1)(p_n-1)}\Bigg]$$

$$H\left[(x_m), y; \frac{tp \; p_1 \ldots p_n}{(1-z_1 \ldots -z_n)}\right] dp\,dp_1 \ldots dp_n$$

$$= (1-z_1 \ldots -z_n)^a \sum_{n=0}^{\infty} \frac{(a)_n \prod_{i=1}^{n} (\alpha_i)_n}{(\alpha+\beta)_n \prod_{i=1}^{n} (\alpha_i+\beta_i)_n}$$

$$F_A\,[a+n, -n, \ldots, -n; \alpha_1, \ldots, \alpha_n; z_1, \ldots, z_n]\; S_n^A\,[(x_n), y]\, t^n, \qquad (2.12)$$

where C_j $(j=1, 2, \ldots, n+1)$ denote Pochhammer's double loops.

3. APPLICATIONS

By specializing various parameters in (2.2), we get (see [12])

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\alpha_1+\beta_1)_n\,(1+\alpha)_n}\; {}_2F_1[-n, \gamma; \alpha_1; z]\; L_n^{(\alpha)}(x)\, t^n$$

$$= (1-z)^{-\gamma}\; F^{1::0;0;0:0;0;1}_{0::1;0;0:0;1;1}\left[\begin{array}{l} \alpha_1 :: -;-;- : -;-;\gamma; \\ - :: \alpha_1+\beta_1;-;- : 1+\alpha; \alpha_1; -; \end{array}\; t, -xt, \frac{z}{z-1}\right], \qquad (3.1)$$

which is presumably a new result. Here, and in what follows, $F(x, y, z)$ denotes a general triple hypergeometric function of Srivastava (*Cf.*, *e.g.*, [3, p. 109, Eq. (3.7.4)]).

Applying the usual conditions, from (2.2), we get (see [12])

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\alpha_1+\beta_1)_n\,(\beta+1)_n}\; {}_2F_1[-n, \gamma ; \alpha_1 ; z]\; P_n^{(\beta,\, \alpha-n)}(x)\, t^n$$

$$= (1-z)^{-\gamma}\, F^{1::0;0;0:0;1;1}_{0::0;0;0:0;1;1}\left[\begin{matrix} \alpha_1 ;: - ; - ; - : - ; 1+\alpha+\beta ; \gamma ; \\ -:: \alpha_1+\beta_1 ; - ; -: - ; 1+\beta ; \alpha ; \end{matrix}\; t,\ \frac{x-1}{2}\, t,\ \frac{z}{z-1}\right]. \qquad (3.2)$$

With suitable parameters, (2.5) gives (see [12])

$$\sum_{n=0}^{\infty} \frac{n!\,(\alpha_2)_n}{(\alpha_1+\beta_1)_n\,(\alpha_2+\beta_2)_n\,(1+\alpha)_n}$$

$$P_n^{(\alpha_1-1,\, \alpha_2-\alpha_1)}(1-2z)\; L_n^{(\alpha)}(x)\, t^n$$

$$= (1-z)^{-\alpha}\; F^{2::0;0;0:0;0;0}_{0::2;0;0:0;1;1}$$

$$\left[\begin{matrix} \alpha_1, \alpha_2 :: \quad - \quad ; - ; - : - ; - ; - ; \\ - :: \alpha_1+\beta_1, \alpha_2+\beta_2 ; - ; - ; - ; 1+\alpha ; \alpha_1 ; \end{matrix}\right.$$

$$\left. \frac{t}{1-z},\ \frac{-xt}{1-z}.\ \frac{-z}{1-z}\right]. \qquad (3.3)$$

Changing F_D to F_1 and $S_n^A[(x_m), y]$ to F_2, (2.10) becomes (see [11])

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n \, (a)_n \, (\mu_1 x_1)^n}{n\,!\,(\alpha+\beta)_n} F_1 \left[-n, \gamma_1, \gamma_2 \, ; \alpha \, ; z_1 \, , z_2 \right]$$

$$. \, F_2 \left[-n \, , - \, , b_2 \, ; 1-a-n, \, c_2 \, ; \, \frac{\nu y}{\mu_1 x_1} \, , \, \frac{-\mu_2 x_2}{\mu_1 x_1} \right] t^n$$

$$= (1-z_1)^{-\gamma_1} \, (1-z_2)^{-\gamma_2} \, F \begin{matrix} 2:1\,;1\,;0\,;0\,;1 \\ 2:0\,;0\,;0\,;1\,;1 \end{matrix} \left[\begin{matrix} (-n:0\,;0\,;0\,;1\,;1); \\ (\alpha+\beta:0\,;0\,;1\,;1\;1); \end{matrix} \right.$$

$$\begin{matrix} (\alpha:1\,;..;1):(\gamma_1:1)\,;(\gamma_2:1)\,;-\,;-\,; & (b_2:1)\,; \\ (\alpha:1,1,0,0,0): \quad - \quad ; \quad - \quad ;-;\,(1-a-n:1)\,; & (c_2:1); \end{matrix}$$

$$\left. \frac{z_1}{z_1-1} \, , \, \frac{z_2}{z_2-1} , \, \nu y t, \; \mu_1 x_1 t, \; \mu_2 x_1 t, \; \mu_2 x_2 t \right] \tag{3.4}$$

Changing $S_n^A \, [\,(x_m), y\,]$ to F_A (2.12) reduces to (see [11])

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n \, [\,(\alpha_n)\,]_n \, (a)_n}{(\alpha+\beta)_n \, [\,(\alpha_n+\beta_n)\,]_n \, n!} F_A \left[a+n, \, -n \, , .. , \, -n \, ; \, \alpha_1 \, , .. , \, a_1 \, ; \right.$$

$$\left. z_1 \, , .. , z_n \right] F_A \left[-n, - \, , b_2 \, , .. , \, b_n \, ; 1-a-n, \, c_2 \, , .. , \, c_m \, ; \right.$$

$$\left. \frac{\nu y}{\mu_1 x_1} \, , \, \frac{-\mu_2 x_2}{\mu_1 x_1} \, , \, \ldots \, , \, \frac{-\mu_m x_m}{\mu_1 x_1} \right] (\mu_1 x_1 \; t)^n$$

$$= (1-z_1 \ldots -z_n)^{-\alpha} \, F \begin{matrix} n+2:0\,;..;0\,;1\,;..;1 \\ n+1:1\,;..;1\,;0\,;1\,;..;1 \end{matrix} \left[\begin{matrix} (\alpha:1;..;1); \\ (\alpha+\beta:0,..,0,1,..,1); \end{matrix} \right.$$

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$$(\alpha_1 : 1,0,\ldots,0,1,0,\ldots,0)\,;\ldots;(\alpha_n : 0,\ldots,0,1,1,1,0,\ldots,0)\,;\,(a : 0,\ldots,1,\ldots,1) :$$

$$(\alpha_1+\beta_1 : 0,\ldots,0,1,1,0,\ldots,0)\,;\ldots;(\alpha_n+\beta_n : 0,\ldots,0,1,0,\ldots,0,1)\,;\quad - :$$

$$-\,;\ldots;-\,;-\,;(b_2\,;1)\,;\ldots;(b_n : 1)\,;$$

$$(\alpha_1\,;1)\,;\ldots;(\alpha_n : 1)\,;-\,;(c_2 : 1)\,;\ldots;(c_n : 1)\,;\frac{z_1}{z_1+\ldots+z_n-1},\ldots,$$

$$\left.\frac{z_n}{z_1+\ldots+z_n-1},\ \frac{\nu y t}{1-z_1-\ldots-z_n},\ \frac{\mu_1 x_1 t}{1-z_1-\ldots-z_n},\ldots,\ \frac{\mu_m x_m t}{1-z_1-\ldots-z_n}\right] \qquad (3.5)$$

REMARK. An immediate advantage of these theorems and their applications is that we may easily obtain various other known as well as new multi-dimensional generating relations .

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EFFECT OF SUSPENDED PARTICLES ON THE ONSET OF BÉNARD CONVECTION IN HYDROMAGNETICS THROUGH POROUS MEDIUM

By

K. C. SHARMA
Department of Mathematics, Himachal University,
Shimla - 171005, India

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ABSTRACT

The effect of suspended particles on the onset of Bénard convection is studied in the presence of a uniform vertical magnetic field through porous medium. It is shown that the effects of suspended particles and medium permeability are destabilizing.

1. INTRODUCTION

A detailed account of the thermal instability of fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given in a treatise by Chandrasekhar [2]. Chandra [1] found that the instability depended on the depth of the layer. A Bénard-type cellular convection with fluid descending at the cell centre was observed when the predicted gradients were imposed, if the layer depth is more than 10 mm. Chandra [1] observed that a convection which was different in character from that in deeper layers occurred at much lower gradients than predicted, if the layer depth was less than 7 mm and called this motion columnar instability. He added an aerosol to mark the flow pattern. Motivated by interest in fluid particle mixtures and columnar instability, Scanlon and Segel [5] considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh – number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles .

The medium has been considered to be nonporous in the abovs studies . Lapwood [3] studied the stability of convective flow in hydrodynamics in a porous medium using Rayleigh's procedure . Wooding [6] considered the Rayleigh instability of a thermal boundary layer in flow through a porous medium . When a fluid slowly percolates through the pores of the rock, the gross effect is represented by Darcy's law which states that the usual viscous term in the equations of fluid motion is replaced by the resistence term $\left(\frac{\mu}{K_1}\right)\vec{V}$, where μ is the viscosity of the fluid, k_1 the permeability of the medium and $\vec{V}$ is the velocity of the fluid .

The thermal instability of fluids through porous medium in the presence of suspended particles may find applications in various problems particularlarly in biomechanics, geophysics and chemical technology . It is therefore the motivation of this study to examine the effects of suspended particles on the onset of Bénard convection in hydromagnetics through porous medium .

2. FORMULATION OF THE PROBLEM

Consider an infinite horizontal gas–particle layer of thickness d bounded by the planes $z=0$ and $z=d$ and flowing through porous medium . This layer is heated from below such that a steady adverse temperature gradient $\beta \left(= \left| \frac{dT}{dz} \right| \right)$ is maintained . The gas is assumed to be infinitely conducting . The system is acted on by a uniform vertical magnetic field $\vec{H}$ (0, 0, H) and a gravity force $\vec{g}$ (0, 0, $-g$) . The particles are assumed to be non-conducting.

The relevant equations and boundary conditions in their non-dimensional form governing the infinitesimal linear instability of the above physical system are given as follows :

$$(D^2-k^2-Bn)\ \left[L_1-L_2 (D^2-k^2) + \frac{L_2}{p} - L_2QD^2 \right] (D^2-k^2)W$$

$$= R (z_0n + B)\, k^2W , \qquad (1)$$

$$W = \frac{\partial^2 W}{\partial z^2} = \left(L_1-L_2\, \nabla^2 + \frac{L_2}{P} \right) \nabla^2 W - L_2 Q \nabla^2 \frac{\partial^2 W}{\partial z^2} = 0 ,$$

$$\text{at } z = 0 \text{ and } z = 1 \qquad (2)$$

where Q is the Chandrasekhar number, p is the permeability of the medium and various other symbols occuring in the above equations have the same meaning as in [5] .

3. DISPERSION RELATION AND DISCUSSION

The principle of the exchange of stabilities is valid for this problem as can be easily demonstrated by the usual Pellew and Southwell [4] technique . The marginal state is thus characterized by $n = 0$ and equation (1) and boundary conditions (2), then become

$$(D^2-k^2)^2 \left[D^2-k^2 + QD^2 - \frac{1}{P} \right] W = - k^2\, Rc\, BW , \qquad (3)$$

$$W = \frac{\partial^2 W}{\partial z^2} = \frac{\partial^4 W}{\partial z^4} = 0 \text{ , at } z=0 \text{ and } z=1 \text{ .} \tag{4}$$

where R_C is the critical Rayleigh number . It can be shown that all the even derivatives of W vanish on the boundaries and characterzing the lowest mods is

$$W = A \, Sin \, \pi z \text{ ,} \tag{5}$$

where A is a constant .

Substituting the solution (5) in equation (3) , we obtain

$$R_C = \frac{(\pi^2 + k^2)^2 \left(\pi^2 + k^2 + \frac{1}{P} + Q\pi^2 \right)}{k^2 (b + 1)} \text{ .} \tag{6}$$

It is clear from equation (6) that the magnetic field has an inhibiting effect on the onset of Bénard convection . If R_C is critical Rayleigh number in the presence of suspended particles and $\overline{R_C}$ is the critical Rayleigh number in the absence of suspended particles, then

$$R_C = \frac{\overline{R_C}}{b+1} \text{ ,} \tag{7}$$

which shows that the effect of suspended particles is to destabilize the layer .

It follows from equation (6) that

$$\frac{dR_c}{db} = - \frac{(\pi^2 + k^2)^2 \left(\pi^2 + k^2 + \frac{1}{P} + Q\pi^2\right)}{k^2 (b + 1)^2}, \qquad (8)$$

which is negative . This re-affirms the destabilising effect of suspended particles .

To find the role of permeability of the medium, we examine the nature of $\frac{dR_c}{dP}$.

It follows from equation (6) that

$$\frac{dR_c}{dP} = - \frac{(\pi^2 + k^2)^2}{k^2 (b+1) P^2}, \qquad (9)$$

which is always negative . The effect of medium permeability is, therefore, to destabilize the layer .

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THERMAL CONVECTION IN A POROUS MEDIUM IN THE PRESENCE OF SUSPENDED PARTICLES

By

R. C. SHARMA
Department of Mathematics, Himachal Pradesh University,
Shimla - 171 005, India

and

J. N. MISRA
Department of Mathematics, Government College,
Bilaspur - 174 001, Himachal Pradesh, India

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ABSTRACT

The effect of suspended particles is considered on thermal convection in a porous medium. Principle of exchange of stabilities is satisfied. The effect of suspended particles is to destabilize the layer. The medium permeability also has a destabilizing effect on the system. The effect of uniform rotation is also considered and is found to stabilize a certain wave-number range in thermal convectian in porous medium, which were unstable in the absence of rotation.

1. INTRODUCTION

A comprehensive account of the thermal convection problem, under varying assumptions of hydrodynamics, has been given by Chandrasekhar [1]. Lapwood [2] has studied the stability of convective flow in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [5] . The gross effect, when the fluid slowly percolates through the pores of the rock, is represented by the Darcy's law .

The fluid may not be absolutely pure but may, instead, be permeated with suspended (or dust) particles , The effect of particle mass and heat capacity on the onset Bénard convection has been considered by Scanlon and Segel [3]. The effect of suspended particles was found to destabilize the layer . In another context. Palaniswamy and Purushotham [4] have studied the stability of shear flow of stratified fluids with fine dust and have found the effect of fine dust to increase the region of instability .

Motivated by interest in fluid-particle mixtures and keeping in mind the geophysical situations occurring below the Earth's surface, we set out to study the effect of suspend particles on thermal convection in a porous medium . The effect of uniform rotation on the problem has also been considered due to the importance of the Coriolis force .

2. DESCRIPTION OF THE PROBLEM AND PERTURBATION EQUATIONS

Here we consider a horizontal fluid - particle layer of depth d unbounded in the horizontal (x, y) directions and bounded by the planes $z=0$ and $z=d$ in a porous medium . This layer is heated from below such that a steady adverse temperature gradient β ($= | dT/dz |$) is maintained . The equations of motion and continuity for the fluid are

$$\frac{\rho}{\epsilon}\left[\frac{\partial \mathbf{u}}{\partial t}+\frac{1}{\epsilon}\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\right]=-\nabla p-g\rho\vec{\lambda}-\frac{\mu}{k_1}\mathbf{u}+\frac{kN}{\epsilon}\left(\mathbf{v}-\mathbf{u}\right), \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 . \qquad (2)$$

Here ρ , μ $(=\rho\nu)$, p and $\mathbf{u}$ (u, v, w) denote respectively the density, the viscosity, the pressure and the velocity of the pure fluid. $\mathbf{v}$ (l, r, s) and N $(\bar{x}, t)$ denote the velocity and number density of the particles . ϵ is the medium porosity, k_1 is the medium permeability, g is the acceleration due to gravity, $\bar{x} = (x, y, z)$, $\vec{\lambda} = (0, 0, 1)$ and $K = 6 \pi \mu \eta$, η being the particle radius , is the Stokes' drag . Assuming uniform particle size, spherical shape and small relative velocities between the fluid and particles, the presence of particles adds an extra force term in the equations of motion (1), proportional to the velocity difference between particles and fluid .

The force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid . The buoyancy force on the particles is negligibly small . Interparticle reactions are ignored for we assume that the distances between particles are quite large compared with their diameter . The equations of motion and continuity for the particles are

$$mN \left[\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\epsilon} \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} \right] = kN \left(\mathbf{u} - \mathbf{v} \right) - \epsilon m N g \vec{\lambda}, \qquad (3)$$

$$\epsilon \frac{\partial N}{\partial t} + \nabla \cdot \left(N \mathbf{v} \right) = 0 , \qquad (4)$$

where mN is the mass of particles per unit volume .

Let C, Cp, T and q denote respectively the heat capacity of fluid at constant pressure, the heat capacity of particles, the temperature and the "effective" thermal conductivity is the conductivity of the clean fluid . If the particles and the fluid are assumed to be in thermal equilibrium, then the heat conduction equation gives

$$\left[\rho C\epsilon + \rho_s C_s (1-\epsilon) \right] \frac{\partial T}{\partial t} + \rho C (\mathbf{u} \cdot \nabla) T + mNC_p \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T$$

$$= q\nabla^2 T , \qquad (5)$$

where ρ_s, C_s are the density and the heat capacity of the solid matrix respectively . The equation of state for the fluid is

$$\rho = \rho_0 \left[1-a (T-T_0) \right] , \qquad (6)$$

where ρ_0 is the mean density of the clean fluid and a is the coefficient of thermal expansion .

The initial state of the system, denoted by subscript 0, is taken to be a quiescent layer with a uniform particle distribution N_0 · The initial state

$\mathbf{u}_0 = 0$, $\mathbf{v}_0 = 0$, $T_0 = -\beta z$, $N_0 =$ constant ,
is an exact solution to the governing equations .

Let $\delta\rho$, δp, N, θ , $\mathbf{u}$ and $\mathbf{v}$ denote, respectively, the perturbations in density ρ_0, pressure p_0, number density N_0, temperature T, fluid velocity (zero initially) and particle velocity (zero initially) . We scale the physical variables using

d, $\frac{d^2}{k}$, $\frac{k}{d}$, $\frac{\rho\nu k}{d^2}$ and βd as the length, time, velocity, pressure and temperature scale factors respectively . Then the linearized dimensionless perturbation equations are

$$N_p^{-1}\ \frac{\partial \mathbf{u}}{\partial t} = -\nabla\,\delta p + N_R\theta\ \vec{\lambda} - P^{-1}\,\mathbf{u} + \omega\,(\mathbf{v}-\mathbf{u})\,, \tag{7}$$

$$\tau\ \frac{\partial \mathbf{v}}{\partial t} = \mathbf{u} - \mathbf{v}\,, \tag{8}$$

$$\nabla\cdot\mathbf{u} = 0\,,\ \frac{\partial M}{\partial t} + \nabla\cdot\mathbf{v} = 0\,, \tag{9}$$

$$E\,(1+h)\ \frac{\partial\theta}{\partial t} = (W+hs) + \nabla^2\theta \tag{10}$$

In writing eq . (7) , use has been made of the Boussinesq equation of state $\delta\rho = -\rho_0 a\theta$. w and s are the vertical fluid and particle velocities , ν is the kinematic viscosity of the fluid, k is the thermal diffusivity , $N_p = \epsilon\nu/k$ is the modified Prandtl number, $N_R = ga\,\beta\,d^4/\nu k$ is the Rayleigh number, $f = mN_0/\rho_0 = \tau w\, N_p$ is the mass fraction, $\omega = KN_0 d^2 / \rho_0\nu\epsilon$, $P = k_1 / d^2$, $\tau = mk / kd^2$, $h = fC_p / C$, $M = N / N_0$ and $E = \epsilon + (1-\epsilon)\,\rho_s C_s / \rho C$.

Let us consider the case of two free surfaces having uniform temperature . The case of two free surfaces , though artificial in nature , enables us to find the analytical solution . The boundary conditions appropriate for the problem are

$$W = \frac{\partial^2 w}{\partial z^2} = \theta = 0 \qquad \text{at } z = 0 \text{ and } 1\,. \tag{11}$$

Eliminating $\mathbf{v}$ and δp the fluid and heat equations give

$$\left(L_1 + \frac{L_2}{P}\right)\nabla^2 w = L_2 N_R\,\nabla_1^2\,\theta\,, \tag{12}$$

$$L_2\left(EH\,\frac{\partial}{\partial t} - \nabla^2\right)\theta = \left(\tau\,\frac{\partial}{\partial t} + H\right)W\,, \tag{13}$$

where

$$L_1 = N_p^{-1}\left(\tau \frac{\partial^2}{\partial t^2} + F\frac{\partial}{\partial t}\right),\ L_2 = \left(\tau\frac{\partial}{\partial t} + 1\right),$$

$$F = f + 1,\ H = h + 1,\ \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

Eliminating θ between eqs. (12) and (13), we obtain

$$\left(L_1 + \frac{L_2}{P}\right)\left(EH\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\omega = N_R\left(\tau\frac{\partial}{\partial t} + H\right)\nabla_1^2\omega \tag{14}$$

Analyze the perturbations in terms of normal modes by seeking solutions whose dependence on x, y and t is given by

$$w = W(z)\exp(i\alpha_x x + i\alpha_y y + nt), \tag{15}$$

where $\alpha = \left(\alpha_x^2 + \alpha_y^2\right)^{1/2}$ is the wavenumber of the disturbance and n is the growth rate.

Equation (14), using (15), becomes.

$$\left(L_1 + \frac{L_2}{P}\right)\left(D^2 - \alpha^2\right)\left(D^2 - \alpha^2 - EHn\right)W = \left(\tau n + H\right)N_R\alpha^2 W, \tag{16}$$

where

$$L_1 = N_p^{-1} \; (\tau n^2 + Fn) , \quad L_2 = (1 + \tau n) \quad \text{and } D = \frac{d}{dz} \; .$$

3. FROOF OF THE PRINCIPLE OF EXCHANGE OF STABILITIES

In this section we shall show that, for the problem under discussion, the principle of exchange of stabilities is valid, *i. e.* n is real and the marginal states are characterized by $n = 0$.

Let $$U = (D^2 - \alpha^2) W , \tag{17}$$

and $$X = \left(L_1 + \frac{L_2}{P} \right) U . \tag{18}$$

The equation satisfied by w, in terms of X, is

$$(D^2 - \alpha^2 - Hn) X - \alpha^2 (\tau n + H) N_R W . \tag{19}$$

Multiply eq. (19) by X^*, the complex conjugate of X and integrate over the range of z. We have

$$\int_0^1 X^* (D^2 - \alpha^2 - Hn) X dz = \alpha^2 \; (\tau n + H) \; N_R \int_0^1 X^* W dz . \tag{20}$$

Integrating by parts and using eqs. (17), (18) and boundary conditions (11), eq. (20) yields

$$I_1 + \left(\alpha^2 + EHn \right) I_2 = \alpha^2 N_R \left(\tau n + H \right) \left(L_1^* + \frac{L_2^*}{P} \right) I_3 , \tag{21}$$

where

$$I_1 = \int_0^1 | DX |^2 dz \; ,$$

$$I_2 = \int_0^1 | x |^2 \, dz \; , \qquad (22)$$

$$I_3 = \int_0^1 (| DW |^2 + \alpha^2 | W |^2) \, dz \; ,$$

which are all positive definite . Putting $n = in_0$, where n_0 is real, into eq . (21) and equating the imaginary parts, we obtain

$$n_0^2 = - \frac{EHI_2 + \alpha^2 N_R (N_p^{-1} FH + \tau h P^{-1}) I_3}{\alpha^2 N_R N_p^{-1} \tau^2 I_3} \; , \qquad (23)$$

$$\text{or} \qquad n_0 = 0 \; . \qquad (24)$$

Since n_0 is real and integrals $I_1 - I_3$ are positive definite, it follows that $n_0 = 0$. Therefore n is real and the principle of exchange of stabilities is valid for the problem under consideration .

4. DISPERSION RELATION AND DISCUSSION

It has been proved above that marginal state is that of stationary convection and the principle of exchange of stabilities is valid . When instability sets in as stationary convection , the marginal state will be characterized by $n = 0$ and eq. (16) reduces to

$$\frac{1}{P} (D^2 - \alpha^2)^2 \; W = H \alpha^2 N_R W \; . \qquad (25)$$

Here we consider the case of two free boundaries . It can be shown that all the even order derivatives of W vanish on the boundaries and hence the proper solution of (25) characterizing the lowest mode is

$$W = W_0 \sin \pi z \; , \qquad (26)$$

where W_0 is a constant .

Substituting the solution (26) in eq . (25), we obtain

$$N_R = \frac{(\pi^2 + \alpha^2)^2}{\alpha^2 HP} \tag{27}$$

Equation (27) yields

$$\frac{dN_R}{dp} = -\frac{(\pi^2 + \alpha^2)^2}{\alpha^2 HP^2} , \tag{28}$$

and

$$\frac{dN_R}{dH} = -\frac{(\pi^2 + \alpha^2)^2}{\alpha^2 PH^2} . \tag{29}$$

The medium permeability has a destabilizing effect on the system, The effect of suspended particles is also to destabilize the layer .

5. EFFECT OF UNIFORM ROTATION

In this section we consider the same problem as described above except that the system is in a state of uniform rotation $\vec{\Omega}\ (0, 0, \Omega)$. The Coriolis force on the particles is neglected . Equations (8) – (10) remain unaltered , Denoting the Taylor number $T_A = 4\Omega^2 d^4 / \nu^2$, the linearized nondimensional perturbation equations of motion for the fluid are

$$N_p^{-1} \frac{\partial u}{\partial x} = -\frac{\partial}{\partial x} \delta p + \omega (l - u) + T_A^{1/2}\ v - \frac{1}{p} u , \tag{30}$$

$$N_p^{-1} \frac{\partial v}{\partial t} = -\frac{\partial}{\partial y} \delta p + \omega (\gamma - v) - T_A^{1/2} u - \frac{1}{P} v , \tag{31}$$

$$N_p^{-1}\ \frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}\ \delta p + \omega\,(\delta - w) + N_R\,\theta - \frac{1}{P}\ w\ . \qquad (32)$$

Eliminating v and δp between eqs. (30) – (32) and (8), we obtain

$$\left(L_1 + \frac{L_2}{P}\right)^2 \nabla^2 w + L_2^2\ T_A \frac{\partial^2 w}{\partial z^2} = L_2\left(L_1 + \frac{L_2}{P}\right) N_R\ \nabla_1^2\ \theta\ . \qquad (33)$$

Eliminating θ between (10) and (33) and using, expressing (15), we obtain

$$\left(D^2 - \alpha^2 - EH\epsilon\right)\left[\left(L_1 + \frac{L_2}{P}\right)^2\left(D^2 - \alpha^2\right) + L_2^2\ T_A D^2\right]W$$

$$= \left(L_1 + \frac{L_2}{P}\right)\alpha^2 N_R\left(\tau\epsilon + H\right)W\ . \qquad (34)$$

For the stationary convection, $n = 0$ and eq. (34) reduces to

$$\left(D^2 - \alpha^2\right)\left[\frac{1}{P^2}\left(D^2 - \alpha^2\right) + T_A D^2\right]W = \frac{1}{P}\ \alpha^2 H N_R\ W\ . \qquad (35)$$

Again considering the case of two free boundaries with constant temperature and using the proper solution (26), eq. (35) yields

$$N_R = \frac{(\pi^2 + \alpha^2)}{\alpha^2 H}\left[\frac{1}{P}\left(\pi^2 + \alpha^2\right) + \pi^2 T_A P\right]. \qquad (36)$$

Equation (36) yields

$$\frac{dN_R}{dP} = \frac{(\pi^2 + \alpha^2)}{\alpha^2 H}\left[-\frac{(\pi^2 + \alpha^2)}{P^2} + T_A \pi^2\right]. \qquad (37)$$

If $T_A > \left(1 + \frac{\alpha^2}{\pi^2}\right) / P^2$, $\frac{dN_R}{dp}$ is positive and if

$T_A < \left(1 + \frac{\alpha^2}{\pi^2}\right) / P^2$, $\frac{dN_R}{dP}$ is negative . So the medium permeability has both stabilizing and destabilizing effects depending on the Taylor number . The rotation, thus, stabilizes a certain wave-number range in thermal convection in a porous medium in the presence of suspendde particles, which were unstable in the absence of rotation .

Equation (36) also yields

$$\frac{dN_R}{dT_A} = \frac{P\pi^2 (\pi^2 + \alpha^2)}{\alpha^2 (1 + h)} \quad , \tag{38}$$

and

$$\frac{dN_R}{dh} = - \frac{(\pi^2 + \alpha^2) \left[\frac{1}{P} \quad (\pi^2 + \alpha^2) + \pi^2 T_A P \right]}{\alpha^2 (1 + h)^2} \; . \tag{39}$$

The rotation has stabilizing effect whereas the suspended particles have destabilizing effect on the system under consideration .

6. DISCUSSION

Owing to great importance and applications in Geophysics, the problem of thermal convection in a porous medium in the presence of suspended particles has been considered in the present paper . As in thermal convection problem, the principle of exchange of stabilities is satisfied for the thermal convection in porous medium in the presence of suspended particles and the oscillatory modes are not allowed . In contrast to the effects of rotation and magnetic field on the thermal convection which introduce oscillatory modes, the presence of suspended particles and medium porosity are unable to bring in overstability . Since the heat capactiy of the pure fluid is supplemented by that of the particles, the effect of suspended particles is to destabilize the layer . The medium permeability also has a destabilizing effect on the system . The rotation stabilizes a certain wave-number range in thermal convection in a porous medium in the presence of suspended particles, which were unstable in the absence of rotation and thus rotation has stabilizing effect on the system .

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ON PLANE WAVES OF SOUND IN A POROUS MEDIUM

By

R. C. SHARMA
Department of Mathematics, Himachal Pradesh University,
Shimla - 171 005, India
and
BAKSHIS SINGH
Department of Mathematics, S. V. S. D. College,
Bhatoli - 140126, P. O. Naya Nangal, Himachal Pradesh, India

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ABSTRACT

The effects of viscosity and thermal conductivity on plane wave propagation through compressible fluid in a rigid, homogeneous and isotropic porous medium are considered. The amplitude of the waves attenuates exponentially as they propagate. This attenuation is increased with increases in viscosity, thermal conductivity and frequency and decreases in the permeability.

1. INTRODUCTION

A detailed account of the propagation of plane waves of sound in the presence of viscosity and heat conduction effects has been given by Lamb [3] and Rayleigh [5]. The action of porous bodies in the absorption of sound for the important though special case of waves in tubes has also been treated by Lamb [3] and Rayleigh [5]. Morse [4] has extended this study and compared his theoretical results of the attenuation with the experimental values. However, he made several assumptions on the particle sizes and frequency ranges.

The problem needs to be studied in its full generality. A re-examination of the effects of viscosity and thermal conductivity on the wave propagation through a compressible fluid in a rigid, homogeneous and isotropic porous medium is certainly called for and is the object of the present paper.

2. PERTURBATION EQUATIONS

Here we consider the plane wave propagation in x-direction through a porous medium. The porous medium is assumed to be rigid, homogeneous and isotropic. Then the equations of motion and continuity governing the flow of viscous, compressible fluid through a porous medium are (Joseph [2], Yih [6]) :

$$\frac{1}{\epsilon}\left(\frac{\partial u}{\partial t}+\frac{1}{\epsilon}\,u\,\frac{\partial u}{\partial z}\right)=-\frac{1}{\rho}\,\frac{\partial p}{\partial x}+\frac{4\mu}{3\rho\epsilon}\,\frac{\partial^2 u}{\partial x^2}-\frac{\mu}{\rho k_1}\,u\,, \tag{1}$$

and

$$\epsilon\,\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0 \tag{2}$$

Here u, p, ρ and μ denote respectively the velocity, pressure, density and viscosity of the fluid wereas ϵ and k_1 stand for medium porosity and medium permeability. For a unit volume of the porous medium, the amount of heat required to produce small changes in the volume V and temperature T (Lamb [3], Joseph [2]) is

$$\delta Q=\frac{\rho_0 C_v(\gamma-1)T_0}{V_0}\,\delta V+\left[\rho_0 C_v\epsilon+\rho_s C_s(1-\epsilon)\right]\delta T\,, \tag{3}$$

where ρ_0, C_v ; ρ_s, C_s stand for the density and the specific heat at constant volume of the fluid and the solid respectively.

$\gamma = \left(\frac{C_p}{C_v} \right)$ denotes the ratio of specific heats of the fluid.

Multiplying (3) by δx and differentiating with respect to t, we obtain the rate at which heat must be supplied to the stratum and this on equating to $K \frac{\partial^2 T}{\partial x^2} \delta x$ gives

$$E \frac{\partial T}{\partial t} + (\gamma - 1) \frac{T_0}{V_0} \frac{\partial V}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} , \tag{4}$$

where K is the 'effective' thermal conductivity,

$\kappa = \left(\frac{K}{\rho_0 C_v} \right)$ is the thermal diffusivity and $E = \epsilon + \frac{(1 - \epsilon) \rho_s C_s}{\rho_0 C_v}$.

The macroscopic heat conduction eq. (4) can also be obtained by an averaging procedure starting from the microscopic scale (inside the pore space) to the macroscopic one (Bear [1]) .

The equation of state is

$$\frac{p}{p_0} = \frac{\rho T}{\rho_0 T_0} \tag{5}$$

Substituting

$$\rho = \rho_0 (1+S) , \quad T=T_0 (1+\eta) , \tag{6a, b}$$

and retaining only the linear terms, eqs. (1), (2), (4) and (5) give

$$\frac{1}{\epsilon} \frac{\partial u}{\partial t} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \frac{4\nu}{3\epsilon} \frac{\partial^2 u}{\partial x^2} - \frac{\nu}{k_1} u , \tag{7}$$

$$\epsilon \frac{\partial s}{\partial t} = - \frac{\partial u}{\partial x} , \tag{8}$$

$$E \frac{\partial \eta}{\partial t} - (\gamma-1) \frac{\partial s}{\partial t} = \kappa \frac{\partial^2 \eta}{\partial x^2} , \tag{9}$$

$$p = p_0 (1 + s + \eta) , \tag{10}$$

where $\nu \; (= \mu / \rho)$ is the kinematic viscosity of the fluid .

Equations (7) to (10) yield

$$\frac{1}{\epsilon} \frac{\partial^2 u}{\partial t^2} = \frac{b^2}{\epsilon} \frac{\partial^2 u}{\partial x^2} - b^2 \frac{\partial^2 \eta}{\partial x \partial t} + \frac{4\nu}{3\epsilon} \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\nu}{k_1} \frac{\partial u}{\partial t} , \tag{11}$$

$$\text{and} \quad E \frac{\partial \eta}{\partial t} + \frac{\gamma - 1}{\epsilon} \frac{\partial u}{\partial x} - x \frac{\partial^2 \eta}{\partial x^2} \quad s \tag{12}$$

where $b^2 \; (= p_0 / \rho_0)$ denotes the square of the Newtonian velocity of sound .

3. DISCUSSION

Assuming that u and η both vary as

$$a \; exp \; (i\sigma t + kx) , \tag{13}$$

where k, σ and a denote the wave number , growth rate and amplitude respectively and eliminating η between (11) and (12) , we obtain the dispersion relation

$$x \left(ib^2 - \frac{4\nu\sigma}{3} \right) k^4 + \left[E\sigma \left(b^2 + \frac{4i\nu\sigma}{3} \right) + x \left(i\sigma^2 + \frac{\nu\epsilon\sigma}{k_1} \right) \right.$$

$$\left. + \, b^2\sigma \left(\gamma - 1 \right) \right] k^2 + E \left(\sigma^3 - \frac{i\nu\epsilon\sigma^2}{k_1} \right) = 0 . \tag{14}$$

Now several cases of interest are derived and discussed .

(i) For fluid continuum ($\epsilon = 1$, $E = 1$, $k_1 \rightarrow \infty$) and in the absence of dissipative effects of viscosity and thermal conductivity

($\nu = x = 0$), eq . (14) gives

$$k = \pm \frac{i\sigma}{c} , \tag{15}$$

where c ($\sqrt{\gamma p_0/\rho_0}$) denotes the velocity of sound .

(ii) For fluid continuum, $\nu = 0$ (nonviscous fluid) and (practically isothermal conditions). eq . (14) yields

$$k = \pm \frac{i\sigma}{b} \tag{16}$$

(iii) For fluid continuum where only dissipation due to viscosity is considered (*i. e.* $x = 0$) , eq . (14) yields

$$k = - \frac{i\sigma}{c} - \frac{2}{3} \frac{\nu\sigma^2}{c^3} , \tag{17}$$

where negative sign has been taken and $\nu\sigma / c^2$ being small, its square and higher powers have been neglected .

Substituting this value of k in (13), we obtain, for the waves propagated in the direction of x–positive .

$$u = Re \left\{ a e^{-x/l} \cdot e^{i\sigma (t - \frac{x}{c})} \right\} = a e^{-x/l} \cos \sigma \left(t - \frac{x}{c} \right) , \tag{18}$$

where Re denotes the real part and $l = \frac{3}{2} \frac{c^3}{\nu\sigma^2}$.

Thus the amplitude of the waves decreases exponentially as x increases and if σ is large the diminution is more rapid . To the first order of $\nu\sigma/c^2$, the wave velocity is unaffected by the dissipation.

(iv) For porous medium where dissipation due to viscosity alone is considered (*i. e.* $x = 0$), eq . (14) gives

$$k = \pm \frac{i\sigma}{b}\left(\frac{E}{\gamma-1+E}\right)^{1/2}\left(1-\frac{i\nu\epsilon}{k_1\sigma}\right)^{1/2}\left(1-\frac{2i\nu\sigma E/3b^2}{\gamma-1+E}\right), \quad (19)$$

square and higher powers of $\nu\sigma / c^2$ being neglected . For large $k_1\sigma$ and taking the lower sign, eq . (19) gives

$$k = -\frac{i\sigma}{b\left(\frac{\gamma-1+E}{E}\right)^{1/2}}\left[1-\frac{\nu^2\epsilon}{3k_1\, b^2\left(\frac{\gamma-1+E}{E}\right)}\right]$$

$$-\frac{\nu}{b\left(\frac{\gamma-1+E}{E}\right)^{1/2}}\left[\frac{\epsilon}{2k_1}+\frac{2\sigma^2}{3b^2\left(\frac{\gamma-1+E}{E}\right)}\right]. \quad (20)$$

Substitute this value of k in (13) and take the real part, we obtain, for the waves propagated in the direction of x-positive,

$$u = ae^{-x/l} \;.\; \cos\sigma\left[t-\frac{x}{b\left(\frac{\gamma-1+E}{E}\right)^{1/2}}\left\{1-\frac{\nu^2\epsilon}{3k_1b^2\left(\frac{\gamma-1+E}{E}\right)}\right\}\right], \quad (21)$$

where

$$l = \frac{b\left(\frac{\gamma-1+E}{E}\right)^{1/2}}{\nu\left[\frac{\epsilon}{2k_1}+\frac{2\sigma^2}{3b^2\left(\frac{\gamma-1+E}{E}\right)}\right]}. \quad (22)$$

Equations (21) and (22) imply that the amplitude of the waves diminish exponentially as they proceed , the diminution being more rapid the greater the values of frequency and fluid viscosity but smaller the value of permeability.

(v) Finally, we consider the case of porous medium where only the effect of thermal conductivity is considered (*i. e.* $\nu = 0$), eq. (14) yields

$$k^2 = -\frac{1}{2ix}\left\{\sigma(\gamma-1+E)+\frac{ix\sigma^2}{b^2}\right\} \pm \frac{1}{2ix}$$

$$\left[\left\{\sigma(\gamma-1+E)+\frac{ix\sigma^2}{b^2}\right\}^2-\frac{4ixE}{b^2}\sigma^3\right]^{1/2} \tag{23}$$

Since the ratio $x\sigma/c^2$ is small for ordinary sound waves, the roots of (23) are approximately

$$k_a^2 = -\frac{\sigma^2}{b^2\dfrac{(\gamma-1+E)}{E}}, \tag{24}$$

and

$$k_b^2 = \frac{i\sigma}{x}(\gamma-1+E). \tag{25}$$

A more accurate value of k_a^2 is

$$k_a^2 = -\frac{\sigma^2}{b^2\dfrac{(\gamma-1+E)}{E}}\left[1-\frac{ix\sigma}{b^2(\gamma-1+E)}\left\{1-\frac{1}{\dfrac{(\gamma-1+E)}{E}}\right\}\right]. \tag{26}$$

Therefore

$$k_a = \pm\left(\frac{i\sigma}{b\sqrt{\dfrac{(\gamma-1+E)}{E}}}+\frac{1}{l}\right). \tag{27}$$

where

$$l = \frac{b^3 \, (\gamma - 1 + E) \left(\dfrac{\gamma - 1 + E}{E} \right)^{1/2}}{\dfrac{1}{2} \, x\sigma^2 \left\{ 1 - \dfrac{1}{\left(\dfrac{\gamma - 1 + E}{E}\right)} \right\}} . \qquad (28)$$

The complete solution for $x > 0$ is then

$$u = a \, e^{i\sigma t + k_a x} + d e^{i\sigma t + k_b x} , \qquad (29)$$

provided k_a, k_b are chosen so that their real parts are negative, a and d being the amplitudes of the respective waves. The second term in the value of u is insignificant in comparison with the first term. Substituting (27) in (29), it is seen that the amplitude of the waves decreases exponentially as they propagate and this attenuation is more rapid the greater the values of frequency and thermal conductivity.

A similar analysis holds for the simultaneous presence of the viscosity and thermal conductivity effects. The effects (decrease in amplitude of the waves) are increased but remain of the same order of magnitude.

Hence we conclude the whole analysis in the following statements. The waves attenuate exponentially as they propagate and the attenuation increases with frequency and the inverse of permeability. The rate of attenuation of the waves also increases with increases in the values of viscosity and thermal conductivity.

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THEORETICAL STUDY OF RELATIONSHIP BETWEEN DERMAL BLOOD FLOW AND SKIN SURFACE TEMPERATURE OF A HUMAN SUBJECT

By

V. P. SAXENA and KAMAL RAJ PARDASANI
School of Studies in Mathematics, Jiwaji University,
Gwalior - 474011, M. P., India

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ABSTRACT

This paper deals with a direct mathematical relationship of skin surface temperature and blood mass flow rate in dermis and subdermal regions of a human body. The investigations are based on the mathematical formulation for heat migration in skin and subcutaneous tissues of a human body. This formulation incorporates the effects of blood mass flow rate, rate of metabolic heat generation and evaporation from the surface at a moderate atmospheric temperature. The variational finite element method has been employed to solve the problem dividing the entire region in four natural compartments, namely, stratum corneum, stratum germinativum, dermis and subcutaneous tissues. Then blood mass flow rate in dermis and subcutaneous part is varied independently. The effect of this variation on the skin surface temperature is studied for suitable numerical values of various physical and physiological quantities thus involved.

1. INTRODUCTION

The heat transfer process through human dermal regions is a complex phenomena due to various reasons. As the core of a human body remains at a fairly uniform temperature, the peripheral region undergoes frequent changes due to variations in environmental conditions. The important factors which account for these changes, are blood flow, metabolic heat generation, water or sweat evaporation from the outer skin surface etc. The region which is known as skin and sub-cutaneous tissues (SST), can broadly be divided into four almost parallel layers. The first two layers are stratum corneum and stratum germinativum and makes the epidermis. The other two layers are dermis and sub-cutaneous tissues. The biological parameters have different properties in these layers. The uppermost one, stratum corneum is made of dead cells and there is no blood flow and metabolic heat generation in this part. The second layer is also free from blood vessels but has some metabolic activity due to presence of some living cells near the lower interface. The next layer (dermis) contains various constituents such as blood capillaries, blood cells, lymphatics, nerves and glands. The sub-cutaneous layer is fully saturated with active cells and blood vessels. The changes taking place in these layers have their effect on the body surface temperature. In fact the body surface temperature is controlled by the internal as well as external parameters. Amongest the internal parameters, the most important one is blood flow. In this paper we investigate a direct relationship between blood mass flow rate in dermis and subcutaneous tissues, and the outer skin surface temperature.

Temperature distribution problems in SST have already been studied theoretically by Saxena ([2] and [3]), Saxena and Arya ([4] and [5]), and Saxena and Bindra [6]. In most of the cases they have used variational finite element method by dividing SST into its natural compartments and taking each one as an element having independent variation of parameters . We use same technique in this paper . Our problem deals with a one dimensional steady state case having heat loss due to radiation, convection and evaporation at the outer surface . Numerical results have been obtained only for a single case of atmospheric temperature and SST structure ;

2. MATHEMATICAL FORMULATION

We employ the following partial differential equation for temperature distribution in SST (see [3])

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial \theta}{\partial x}\right) + MC_b(\theta_b - \theta) + S \tag{1}$$

along with the boundary condition

$$K\frac{\partial \theta}{\partial n} = h(\theta - \theta_a) + LE \tag{2}$$

where

θ = temperature at distance x from the outer skin surface,

ρ = density of the tissue,

c = specific heat of the tissue,

K = thermal conductivity of the tissue,

M = blood mass flow rate,

c_b = specific heat of blood,

θ_b = temperature of blood when it enters the tissue element

S = rate of metabolic heat generation,

t = time,

θ_a = atmospheric temperature,

h = coefficient of convection and radiation,

L = Latent heat of evaporation,

E = rate of evaporation

x = perpendicular distance from the outer skin surface .

n = normal to skin surface .

The terms on the right of equation (1) are respectively Fickian diffusion, perfusion due to blood flow, and the rate of metabolic heat generation . Here M and S may be functions of position and temperature . The equations (1) and (2) when written in one dimensional steady state case can be transformed into Euler–Lagrange's variational form and written as

$$I = \frac{1}{2}\int_0^d \left[K\left(\frac{\partial\theta}{\partial x}\right)^2 + Mc_b\,(\theta_b - \theta)^2 - 2S\theta \right] dx$$

$$+ \frac{h\,(\theta - \theta_a)^2}{2} + LE\theta \tag{3}$$

where d is thickness of the entire region .

We divide the whole region into four elements according to its structure . The epidermis is divided into two elements, The dermis is taken as third element and subdermal tissues as fourth . We assign values $\theta_r\,(r = 0, 1, 2, 3, 4,)$ to θ at interfaces at distances $a_r\,(r = 0, 1, 2, 3, 4)$. Thus θ_1, θ_2 and θ_3 are internal points on the temperature profile curve with θ_0 and θ_4 as end points . Here we assume θ_0 (skin surface temperature) to be unknown and $\theta_4 = \theta_b$ (blood temperature = body core temperature) Let $\theta^{(i)}$ denotes $\theta\,(x)$

for $a_{i-1} < x < a_i$ and I_1, I_2, I_3 and I_4 are the values of I in the four elements, then

$$I = \sum_{i=1}^{4} I_i \tag{4}$$

Let $K_i(x)$, $M_i(x)$, and $S_i(x)$ are the values of K, M and S in the i th element. Then taking linear values of $Q^{(i)}$ with respect to x in terms of nodal temperatures θ_r ($r = 0, 1, 2, 3, 4$) in (3) we get the following equations :

$$\left.\begin{aligned}
I_1 &= \frac{B_1^2}{2} K^{(1)} + \frac{h}{2}\,[\,\theta_0 - \theta_a\,]^2 + LE\theta_0 \\
I_2 &= \frac{B_2^2\ K^{(2)}}{2} - A_2\, S^{(2)} - B_2\, N^{(2)} \\
I_3 &= \frac{B_3^2\ K^{(3)}}{2} + \frac{C_b\,(\,\theta_b - A_3\,)^2\, M^{(3)}}{2} \\
&\quad - B_3\, C_b\,(\,\theta_b - A_3\,)\, P^{(3)} + \frac{C_b\, B_3^2\ Q^{(3)}}{2} - A_3 S^{(3)} - B_3 N^{(3)} \\
I_4 &= \frac{B_4^2\, K^{(4)}}{2} + \frac{C_b\,(\,\theta_b - A_4\,)^2\, M^{(4)}}{2} \\
&\quad - B_4 C_b\,(\,\theta_b - A_4\,)\, P^{(4)} + \frac{C_b\, B_4^2\ Q^{(4)}}{2} - A_4 S^{(4)} - B_4 N^{(4)}
\end{aligned}\right| \tag{5}$$

where

$$\left.\begin{aligned} B_i &= \frac{\theta_i - \theta_{i-1}}{u_i} \\ A_i &= \frac{a_i \theta_{i-1} - a_{i-1}\theta_i}{u_i} \\ u_i &= a_i - a_{i-1} \\ K^{(i)} &= \int_{a_{i-1}}^{a_i} K_i(x)\,dx \end{aligned}\right| \qquad i = 1\,(1)\,4$$

$$\left.\begin{aligned} S^{(i)} &= \int_{a_{i-1}}^{a_i} S_i(x)\,dx \\ N^{(i)} &= \int_{a_{i-1}}^{a_i} x\,S_i(x)\,dx \end{aligned}\right| \qquad i = 2\,(1)\,4$$

$$\left.\begin{aligned} M^{(i)} &= \int_{a_{i-1}}^{a_i} M_i(x)\,dx \\ P^{(i)} &= \int_{a_{i-1}}^{a_i} x\,M_i(x)\,dx \\ Q^{(i)} &= \int_{a_{i-1}}^{a_i} x^2\,M_i(x)\,dx \end{aligned}\right| \qquad i = 3,\ 4$$

Now extremizing I with respect to θ_0, θ_1, θ_2 and θ_3 we get the following system of algebraic equations :

$$\left.\begin{aligned} &(R_1 + h)\,\theta_0 - R_1\theta_1 = D_1 \\ &- R_1\theta_0 + (R_1 + R_2)\,\theta_1 - R_2\,\theta_2 = D_2 \\ &- R_2\theta_1 + R_3\theta_2 + R_4\theta_3 = D_3 + F_1\theta_b \\ &R_4\theta_2 + R_5\theta_3 = D_4 + F_2\theta_b \end{aligned}\right| \qquad (6)$$

where

$$R_1 = \frac{k^{(1)}}{a_1}\ , \quad R_2 = \frac{k^{(2)}}{u_2}$$

$$R_3 = \frac{k^{(2)}}{u_2^2} + \frac{k^{(3)}}{u_3^2} + \frac{c_b\ M^{(3)}\ a_3^2}{u_3^2} - \frac{2c_b P^{(3)} a_3}{u_3^2} + \frac{c_b Q^{(3)}}{u_3^2}$$

$$R_4 = -\frac{k^{(3)}}{u_3^2} - \frac{c_b M^{(3)} a_2 a_3}{u_3^2} + \frac{c_b P^{(3)}\ (a_3+a_2)}{u_3^2} - \frac{c_b Q^{(3)}}{u_3^2}$$

$$R_5 = \frac{k^{(3)}}{u_3^2} + \frac{c_b M^{(3)} a_2^2}{u_3^2} - \frac{2a_2 c_b P^{(3)}}{u_3^2} + \frac{c_b Q^{(3)}}{u_3^2} + \frac{k^{(4)}}{u_4^2}$$

$$+ \frac{c_b M^{(4)}\ a_4^2}{u_4^2} - \frac{2c_b P^{(4)} a_4}{u_4^2} + \frac{c_b Q^{(4)}}{u_4^2}$$

$$D_1 = h\theta_a - LE\ , \ D_2 = \frac{S^{(2)} a_2}{u_2} - \frac{N^{(2)}}{u_2}$$

$$D_3 = \frac{N}{u_2} - \frac{S^{(2)} a_1}{u_2} + \frac{S^{(3)} a_3}{u_3} - \frac{N^{(3)}}{u_3}$$

$$D_4 = \frac{-S^{(3)} a_4}{u_3} + \frac{N^{(3)}}{u_3} + \frac{S^{(4)} a_4}{u_4}$$

$$F_1 = \frac{c_b M^{(3)} a^3}{u_3} - \frac{c_b P^{(3)}}{u_3}$$

$$F_2 = \frac{c_b P^{(3)}}{u^3} - \frac{c_b M^{(3)} a_2}{u_3} + \frac{K^{(4)}}{u_4^2} + \frac{c_b Q^{(4)}}{u_4^2} + \frac{c_b M^{(4)} a_4^2}{u_4^2} - \frac{2c_b P^{(4)} a_4}{u_4^2}$$

Solving the above system of algebraic equations (6) we have

$$\theta_0 = \frac{D_1G_1 + R_1R_2C_1\,[\,R_5D_3 - R_4D_4 + V_1\theta_b\,] + R_1V_3V_4}{C_1G_1} \tag{7}$$

$$\theta_1 = \frac{R_2C_1\,[\,R_5D_3 - R_4D_4 + V_1\theta_b\,] + V_3V_4}{G_1} \tag{8}$$

$$\theta_2 = \frac{V_2\,[\,R_5D_3 - R_4D_4 + V_1\theta_b\,] + V_4R_2R_5}{G_1} \tag{9}$$

$$\theta_3 = \frac{(\,D_4 + F_2\theta_b\,)\,G_1 - R_4V_2\,[\,R_5D_3 - R_4D_4 + V_1\theta_b\,] + R_4V_4R_2R_5}{G_1R_5} \tag{10}$$

where

$C_1 = R_1 + h,$ $\qquad V_1 = R_5F_1 - R_4\,F_2$

$V_2 = R_2C_1 + hR_1\,,$ $\qquad V_3 = R_3R_5 - R_4^2$

$V_4 = D_2C_1 + D_1R_1\,,$ $\qquad G_1 = V_2V_3 - R_2^2\mathrm{R}_5C_1$

3. A SPECIAL CASE

Assuming that the variation of $K_i^{(x)}$, $M_i^{(x)}$ and $S_i^{(x)}$ is insignificant with respect to x and taking

$K_i\,(x) = \mathbf{K}_i$ (constant) for $i = 1\,(1)\,4$

$M_i\,(x) = \mathbf{M}_i$ (constant) for $i = 1\,(1)\,4$

$S_i\,(x) = \mathbf{S}_i$ (constant) for $i = 1\,(1)\,4$

We have

$$\theta_0 = \frac{W_1m^2 + W_2m + b_{14}}{W_3m^2 + \mathrm{W}_4m + b_{19}} \tag{11}$$

where

$$W_1 = \lambda^2 u_3^2 \, b_{10} + \lambda u_3 u_4 b_{11}$$

$$W_2 = \lambda u_3 b_{12} + u_4 b_{13}$$

$$W_3 = \lambda^2 u_3^2 b_{15} + \lambda u_3 u_4 b_{16}$$

$$W_4 = \lambda u_3 b_{17} + u_4 b_{18}$$

$$\mathbf{M} = \lambda \mathbf{M}_4 \, , \; M_3 c_b = \lambda \mathbf{M}_4 c_b = \lambda m$$

$$m = \mathbf{M} c_b$$

$$b_1 = \frac{(\mathbf{S}_3 u_3 - \mathbf{S}_4 u_4)}{12} G_2 \quad , \quad b_2 = \frac{\mathbf{S}_3 u_3 G_2}{6}$$

$$b_3 = \frac{\mathbf{S}_3 u_3 G_2 (2Y_3 + Y_4) + \mathbf{S}_4 u_4 G_2 Y_3}{2}$$

$$b_4 = R_1 \left(\frac{R_2}{3} + Y_3 + \frac{Y_4}{3} \right) V_4 \, , \;\; b_5 = \left(\frac{R_2}{3} + \frac{Y_3}{3} \right) R_1 V_4$$

$$b_6 = R_1 R_2 (Y + Y_4) V_4 + R_1 y_3 Y_4 V_4$$

$$b_7 = \left\{ V_2 \left(\frac{R_2}{3} + \frac{Y_3}{3} + \frac{Y_4}{3} \right) - \frac{R_2^2 C_1}{3} \right\} D_1$$

$$b_8 = \left\{ V_2 \left(\frac{R_2}{3} + \frac{Y_3}{3} \right) - \frac{R_2^2 C_1}{3} \right\} D_1$$

$$b_9 = D_1 R_2 (Y_3 + Y_4) (V_2 - R_2 C_1) + D_1 V_2 Y_3 Y_4$$

$$b_{10} = \frac{R_1 V_4 + D_1 V_2 + G_2 \theta_b}{12} \, , \qquad b_{11} = \frac{R_1 V_4 + D_1 V_2 + G_2 \theta_b}{9}$$

$$b_{12} = b_1 + b_4 + b_7 + \left\{ \frac{D_2}{3} + \left(Y_3 + \frac{Y_4}{3} \right) \theta_b \right\} G_2$$

$$b_{13} = b_2 + b_5 + b_8 + G_2 \left(\frac{D_2}{3} + Y_3 \theta_b \right)$$

$$b_{14} = b_3 + b_6 + b_9 + G_2 \{ D_2 (Y_3 + Y_4) + \theta_b Y_3 Y_4 \}$$

$$b_{15} = \frac{C_1 V_2}{12} \quad , \quad b_{16} = \frac{C_1 V_2}{9}$$

$$b_{17} = C_1 \left\{ V_2 \left(\frac{R_2}{3} + Y_3 + \frac{Y_4}{3} \right) - \frac{R_2^2 C_1}{3} \right\}$$

$$b_{18} = C_1 \left\{ V_2 \left(\frac{R_2}{3} + \frac{Y_3}{3} \right) - \frac{R_2^2 C_1}{3} \right\}$$

$$b_{19} = C_1 R_2 (Y_3 + Y_4) (V_2 - R_2 C_1) + C_1 V_2 Y_3 Y_4$$

$$G_2 = R_1 R_2 C_1 \quad , \quad Y_3 = \frac{\mathbf{K}_3}{u_3}$$

$$Y_4 = \frac{\mathbf{K}_4}{u_4}$$

We use following values for numerical calculations :

$a_0 = 0.0$, $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.6$, $a_4 = 1.1$ cm.

$K_1 = 0.030$ cal / cm–min deg C

$K_2 = 0\ 030$ cal / cm–min deg C

$K_3 = 0.045$ cal / cm–min deg C

$K_4 = 0.060$ cal / cm–min deg C

$h = 0.009$ cal / cm^2-min deg C

$L = 579$ cal / gm

$\theta_a = 23^0 C$

$E = 0.24$, 0.48 and 0.72×10^{-3} gm / cm^2-min

$\theta_b = 27^0 C$

$s = 0.018$ cal / cm^3-min

$S_1 = 0$, $S_2 = 0.0012$, $S_3 = 0.0102$ and $S_4 = 0.018$ cal/cm^3-min.

$m = M_4 c_b = 0.018$ cal / cm^3-min deg C .

4. CONCLUSION

The graphs have been plotted between outer skin surface temperature θ_0 and blood flow rate in dermis and subcutaneous tissues for different values of evaporation at the skin surface . The lower curve ($\lambda = 0$) gives an idea of variation in θ_0 when there is negligible blood flow in dermis due to construction of capillaries . The second curve ($\lambda = \frac{1}{2}$) indicates change in θ_0 when the blood flow rate is half of that in subcutaneous tissues . While the upper most curve ($\lambda = 1$) indicates variation in θ_0 when there is cent percent transfer of blood to this middle layer .

As shown in Fig. 1 all the curves are close to each other, but the gap between the curves [see Fig. 2,3] increases with the increase in rate of evaporation at the skin surface . In all the three figures, the curves diverge from a point at $m = 0$. This divergence seems to reverse for $m = 0.02$ in Fig. 1, for $m = 0.025$ in Fig. 2 and for $m = 0.0475$ in Fig. 3 (which is not visible in the figure) .

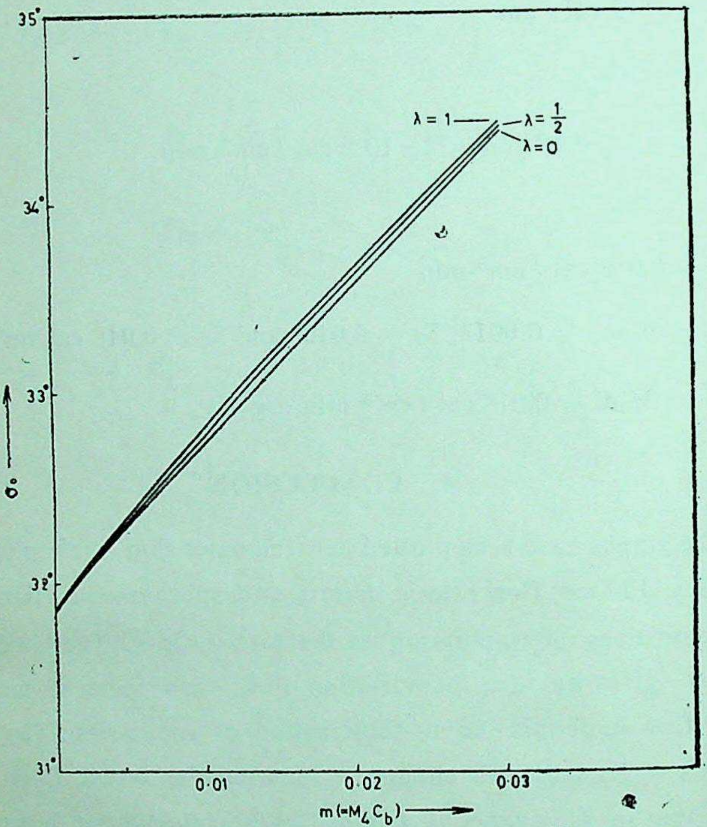

Fig.1:– Graph between σ_0 and m for $\lambda = 0, \frac{1}{2}$ and 1 ($\lambda = \frac{M_3}{M_4}$) and $E = 0\ 24 \times 10^{-3}$ gm/cm^2–min

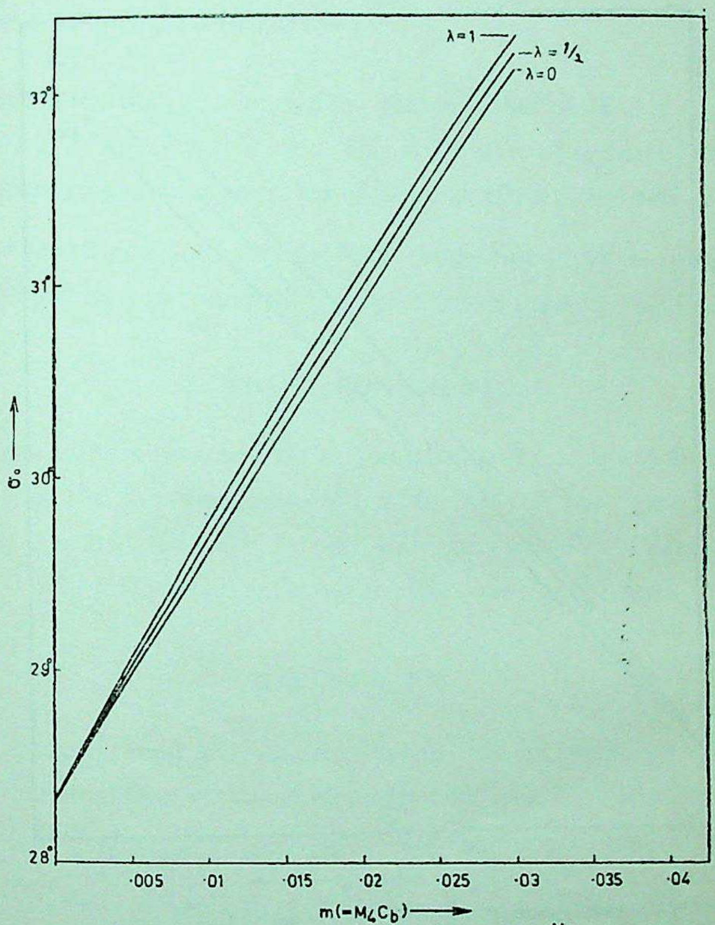

Fig. 2:- Graph between σ_0 and m for $\lambda = 0, \frac{1}{2}$ and 1 ($\lambda = \frac{M_3}{M_4}$) and

$E = 0\ 48 \times 10^{-3}$ gm/cm^2-min

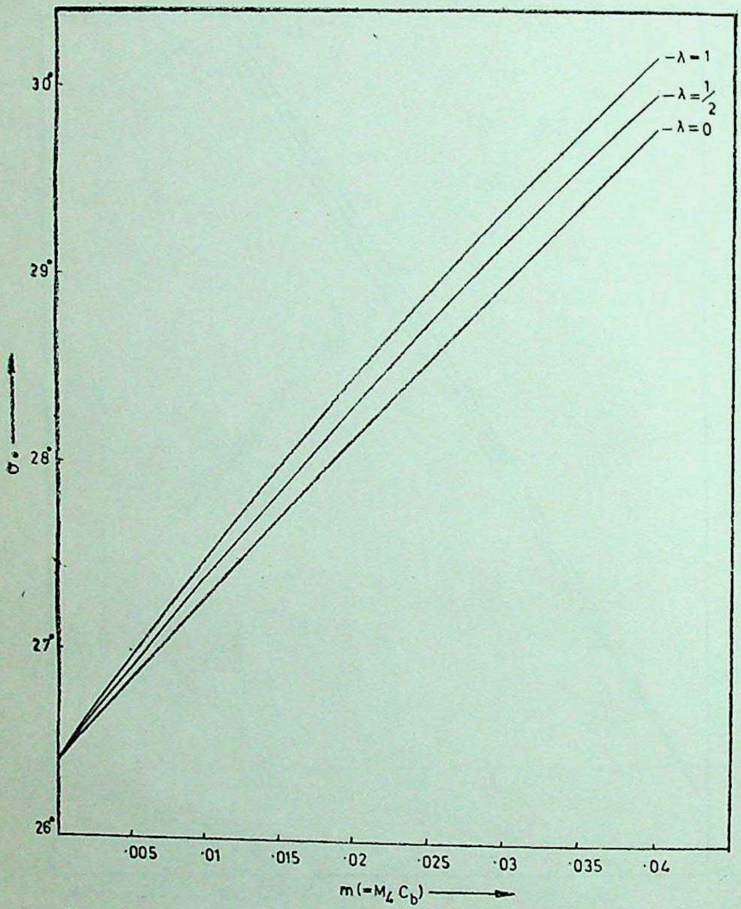

Fig 3:— Graph between σ_0 and m for $\lambda = 0$, $\frac{1}{2}$ and 1 ($\lambda = \frac{M_3}{M_4}$) and $E = 0 \cdot 72 \times 10^{-3}$ gm/cm^2–min

From the above observations we make three conclusions :

1. For higher values of m change in blood flow doesnot have much effect on the skin surface temperature which is in conformity with the results derived by Saxena [3] .

2. At higher rates of evaporation, the contribution of heat transport due to blood flow is more . This is because of increase in temperature gradient due to increased cooling effect at the surface .

3. At zero blood flow rates the heat conduction plays an important role which is in conformity with the tneory used by Perl [1] .

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ON THE MULTIDIMENSIOAL LAPLACE TRANSFORM AND ITS APPLICATIONS

By

RAJNI GUPTA
Department of Mathematics, University of Rajasthan,
Jaipur - 302004, India

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ABSTRACT

In this paper, first we establish an interesting theorem for the multidimensional Laplace transform. Next we obtain a multiple infinite integral involving the product of H-functions of one and several variables having different arguments with the help of our theorem. The main theorem and the integral are quite general in nature and yield a number of (known or new) results as particular cases.

1. INTRODUCTION AND NOTATIONS

The multidimensional Laplace transform is defined by the following integral equation [1] :

$$g(s_1,\ldots,s_r) = (s_1 \ldots s_r) \int_0^\infty \cdots \int_0^\infty exp\left\{ - \sum_{i=1}^{r} s_i x_i \right\} f(x_1,\ldots,x_r)\, dx_1 \ldots dx_r \qquad ..\,(1.1)$$

where $\mathrm{Re}(s_i) > 0$, $\forall i \in \{1,\ldots,r\}$ and the function $f(x_1,\ldots,x_r)$ is so chosen that the above integral is absolutely convergent. As usual, we shall represent (1.1) by the following notation :

$$g(s_1, \ldots, s_r) = L\{f(x_1, \ldots, x_r); s_1, \ldots, s_r\} \qquad \ldots (1.2)$$

The analogue of the well-known Parseval-Goldstein theorem (concerning the Laplace transform) for the transform (1.1) is as follows:

If

$$g_1(s_1, \ldots, s_r) = L\{f_1(x_1, \ldots, x_r); s_1, \ldots, s_r\}$$

and

$$g_2(s_1, \ldots, s_r) = L\{f_2(x_1, \ldots, x_r); s_1, \ldots, s_r\},$$

then

$$\int_0^\infty \cdots \int_0^\infty g_1(x_1; \ldots, x_r)\, f_2(x_1, \ldots, x_r)\, \frac{dx_1}{x_1} \cdot \frac{dx_r}{x_r}$$

$$= \int_0^\infty \cdot \int_0^\infty g_2(x_1, \ldots, x_r)\, f_1(x_1, \ldots, x_r)\, \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} \qquad (1.3)$$

provided that the integrals involved exist.

The H-function of several variables occurring in this peper is a special case of the general H-function of several variables introduced and studied earlier by Srivastava and Panda (cf. [7] and [8]) .

1

The parameters of this function will be displayed in the familiar contracted notation ([6] , *p.* 251 , *Eq:* (*C*.1)) . The details of this function can be found in the papers and the book referred to above .

2. MAIN THEOREM

Under the hypothesis (1.2) , *let*

$$h(s_1, \ldots, s_r) = L\left\{\left(\prod_{i=1}^{r}\left(x_i^{-\rho_i}\right)\right) H_1\left[y_1 x_1^{-\omega_1}, \ldots, y_r x_r^{-\omega_r}\right] f\left(x_1, \ldots, x_r\right);\ s_1, \ldots, s_r\right\}. \qquad (2.1)$$

Then

$$h(s_1,\ldots,s_r) = (s_1 \ldots s_r)\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ x_i^{\rho_i - 1}(x_i + s_i)^{-1} \right\}$$

$$g(x_1 + s_1, \ldots, x_r + s_r)$$

$$. H^{O,O:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1+1;\ldots;p_r,q_r+1}\left[\begin{array}{c|l} y_1 x_1^{\omega_1} & (a_j; a_j', \ldots; a_j^{(r)})_{1,p} : \\ \vdots & \\ y_r x_r^{w_r} & (b_j; \beta_j', \ldots, \beta_j^{(r)})_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{l} (c_j', \gamma_j')_{1,p_1}; \quad .; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (d_j', \delta_j')_{1,q_1}, (1-\rho_1, \omega_1); \ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r}, (1-\rho_r, \omega_r) \end{array}\right]$$

$$dx_1 \ldots dx_r \qquad .. (2.2)$$

provided that

(i) *the n-dimensional Laplace transforms of* $| g(x_1, \ldots, x_r) |$ *and*

$$\left| x_1^{-\rho_1} \ldots x_r^{-\rho_r} \; g(x_1, \ldots, x_r) \, H_1 \left[y_1 x_1^{-\omega_1}, \ldots, y_r x_r^{-\omega_r} \right] \right| \textit{exist};$$

(ii) $U_i - \omega_i > 0, \; | \arg y_i | < \frac{1}{2}(U_i - \omega_i)\pi$,

where

$$U_i = -\sum_{j=1}^{p} \alpha_i^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)}$$

$$+ \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)}, \; \forall \, i \in \{1, \ldots, r\};$$

(iii) $\operatorname{Re}(s_i) > 0, \; \omega_i > 0, \; \operatorname{Re}(\rho_i) + \omega_i \min\limits_{1 \leqslant j \leqslant m_i}$

$$[\operatorname{Re}(d_j^{(i)} / \delta_j^{(i)})] > 0, \; \forall \, i \in \{1, \ldots, r\};$$

(iv) *The multiple integral on the right-hand side of (2.2) is absolutely convergent.*

Proof. We have

$$(s_1 \ldots s_r)\,(s_1 + a_1)^{-1} \ldots (s_r + a_1)^{-1}\, g(s_1 + a_1, \ldots s_r + a_r)$$

$$= L\left\{ exp\left(- \sum_{i=1}^{r} a_i x_i \right) f(x_1, \ldots, x_r)\,;\, s_1, \ldots, s_r \right\} \qquad \ldots (2.3)$$

Also, by virtue of (1.1) and ([6], *p.* 251, *Eq.* (*C*.1)), on interchanging the $(\xi_1, \ldots, \xi_r)$ — and $(x_1, \ldots, x_r)$ — integrals and evaluating the $(x_1, \ldots, x_r)$ — integral with the help of a known result ([8], p. 126, Eq (3.8)), we get

$$(s_1^{-\rho_1+1} \ldots s_r^{-\rho_r+1}\, H_1\left[y_1 s_1^{-\omega_1}, \ldots, y_r s_r^{-\omega_r} \right]$$

$$= L\Big\{ x_1^{\rho_1-1} \ldots x_r^{\rho_r-1}$$

$$. H^{O,\,O\,:\,m_1,\,n_1\,;\,\ldots\,;\,m_r,\,n_r}_{p,\,q\,:\,p_1,\,q_1+1\,;\,\ldots\,;\,p_r,\,q+1} \left[\begin{array}{c|l} y_1 x_1^{-\omega_1} & (a_j\,; \alpha_j', \ldots, \alpha_j^{(r)})\,1.p : \\ \vdots & \\ y_r x_r^{\omega_r} & b_j, \beta_j', \ldots, \beta_j^{(r)})\,1, q : \end{array} \right.$$

$$\left. \begin{array}{l} (c_j', \gamma_j')_{1,p_1}\,; \ldots ;\,(c_j^{(r)}, \gamma_j^{(r)})_{1.p_r} \\ (d_j', \delta_j')_{1,q_1}, (1-\rho_1, \omega_1)\,; \;\ldots\,; (d_j^{(r)}\,; \delta_j^{(r)})_{1,q_r}, (1-\rho_r, \omega_r) \end{array} \right]$$

$$; s_1, \ldots, s_r \Big\} \qquad \ldots (2.4)$$

where the sets of conditions (ii) and (iii) mentioned with the main theorem are satisfied.

Now applying (1.3) for the transform pairs (2.3) and (2.4) and interpreting the left-hand side of the result thus obtained with the help of (2.1), we easily arrive at the result (2.2).

3. SPECIAL CASES

The main theorem established here involves the *H*-function of several variables, which is one of the most general functions introduced so far and includes almost all named special functions of one or more variables. Thus, with the help of results given by R. S. Garg ([2], *pp.* 57-61) and Srivastava, Gupta and Goyal ([6], *pp*. 18-19), one can easily obtain a number of new and interesting theorems by suitably specializing the parameters of the *H*-functions of several variables. To illustrate this, we give below two interesting special cases of the main theorem.

If in the main result we take $p = q = d_j^{(i)} = 0$, $n_i = m_i = p_i = q_i = \gamma_1^{(i)} = \delta_1^{(i)} = \varpi_i = 1$ and $c_1^{(i)} = 1 - \alpha_i \, (i = 1, \ldots, r)$, we get the following corollary :

Corollary 1. *For the functions* $f(x_1, \ldots, x_r)$ *and* $g(x_1, \ldots, x_r)$ *related by* (1.2), *let*

$$h(s_1, \ldots, s_r) = L\left\{ \prod_{i=1}^{r} \left(\Gamma(\alpha_i)\, x_i^{-\rho_i + \alpha_i} (x_i + y_i)^{-\alpha_i} \right) f(x_1, \ldots, x_r) \,;\, s_1, \ldots, s_r \right\} \qquad \ldots (3.1)$$

Then

$$h(s_1, \ldots, s_r) = (s_1 \ldots s_r) \frac{\Gamma(\alpha_1) \ldots \Gamma(\alpha_r)}{\Gamma(\rho_1) \ldots \Gamma(\rho_r)} \int_0^\infty \ldots \int_0^\infty$$

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ x_i^{\rho_i - 1} (x_i + s_i)^{-1} \; {}_1F_1 \, [\alpha_i ; \rho_i ; - y_i x_i] \right\}$$

$$g (x_1 + s_1 , \ldots , x_r + s_r) \, dx_1 \ldots dx_r , \qquad \ldots (3.2)$$

provided that the Laplace transforms of $| \, g (x_1 , \ldots , x_r) \, |$ *and*

$\left| \prod_{i=1}^{r} \left\{ x_i^{-\rho_i + \alpha_i} (x_i + y_i)^{-\alpha_i} \right\} g (x_1 , \ldots , x_r) \right|$ *exist , and the multiple integral on the right-hand side of (3.2) is absolutely convergent .*

Next, putting $p = q = 0$ and $r = 1$ in the main theorem, we get the one-variable case given by

Corollary 2. *If*

$$g (s) = L \{ f(x) \, ; s \} \qquad \ldots (3.3)$$

and

$$h (s) = L \left\{ x^{-\rho} f (x) \; H_{p,q}^{m,n} \left[y x^{-\omega} \left| \begin{matrix} (c_j , \gamma_j)_{1,p} \\ (d_j , \delta_j)_{1,q} \end{matrix} \right. \right] ; s \right] , \qquad \ldots (3.4)$$

then

$$h (s) = s \int_0^\infty x^{\rho - 1} (x + s)^{-1} g (x + s) \; H_{p \, q+1}^{m,n} \left[y x^{\omega} \left| \begin{matrix} (c_j , \gamma_j)_{1,p} \\ (d_j , \delta_j)_{1,q} , (1 - \rho , \omega) \end{matrix} \right. \right] dx \qquad \ldots (3.5)$$

valid under conditions derivable from those for the main theorem .

The theorems established earlier by Saxena ([5], *p*. 230 *Eq*. (2)), Maloo ([4], *p*. 844, *Eq*. (8)) and several others follow as its particular cases, if we reduce the *H*-function involved in corollary 2 to the appropriate special functions occurring in these theorems .

4. APPLICATIONS

Now we shall obtain a very general and interesting integral with the help of the main theorem . Taking,

(4.1) $f(x_1, \ldots, x_r) =$

$$= \prod_{i=1}^{r} \left\{ x_i^{\lambda_i-1} H^{M_i,0}_{P_i,O_i+1}\left[b_i x_i^{t_i} \middle| \begin{matrix} (e_j^{(i)}, E_j^{(i)})_{1,P_i} \\ (f_j^{(i)}, F_j^{(i)})_{1,Q_i}, (1-\lambda_i, t_i) \end{matrix} \right] \right.$$

in (1.2) , we get with the help of a known result ([6] , *p*. 16, *Eq*. (2.4.2)) :

(4.2) $g(s_1, \ldots, s_r) =$

$$= \prod_{i=1}^{r} \left\{ s_i^{-\lambda_i+1} H^{M_i,0}_{P_i,Q_i}\left[b_i s_i^{-t_i} \middle| \begin{matrix} (e_j^{(i)}, E_j^{(i)})_{1,P_i} \\ (f_j^{(i)}, F_j^{(i)})_{1,Q_i} \end{matrix} \right] \right.$$

where

$$U^{(i)} = \sum_{j=1}^{M_i} F_j^{(i)} - \sum_{j=M_i+1}^{Q_i} F_j^{(i)} - \sum_{j=1}^{P_i} E_j^{(i)} - t_i > 0,$$

$$|\arg(b_i)| < \tfrac{1}{2} U^{(i)} \pi, \text{ and } \min_{1 \leqslant i \leqslant r} \{ Re(s_i), Re(\lambda_i), t_i \} > 0.$$

Making use of (2.1) , (4.1) and (1.1) , we easily get

(4 3) $$h(s_1, \ldots, s_r) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ s_i x_i^{\lambda_i-\rho_i-1} e^{-s_i x_i} H^{M_i,0}_{P_i,Q_i+1} \right.$$

$$\left[b_i x_i^{t_i} \middle| \begin{matrix} (e_j^{(i)}, E_j^{(i)})_{1,P_i} \\ (f_j^{(i)}, F_j^{(i)})_{1,Q_i}, (1-\lambda_i, t_i) \end{matrix} \right] H_1\left[y_1 x_1^{-\omega_1}, \ldots, y_r x_r^{-\omega_r} \right]$$

$dx_1 \ldots dx_r$

To evaluate the multiple integral on the right-hand side of (4.3), write out the exponential series. interchange the order of integration and summation (which is justified under the conditions stated below), evaluate the ($x_1, \ldots, x_r$) — integral, so obtained with the help of a known result ([2], p. 83). We thus obtain

$$(4{\cdot}4) \quad h(s_1, \ldots, s_r) =$$

$$= \left(\frac{s_1 \ldots s_r}{t_1 \ldots t_r} \right) \prod_{i=1}^{r} \left\{ b_i^{(\rho_i - \lambda_i)/t_i} \sum_{k_i=0}^{\infty} \frac{(-s_i b_i^{-1/t_i})^{k_i}}{k_i\,!} \right\}$$

$$. H^{0,0\,:\, m_1 + M_1, n_1 ; \ldots ; m_r + M_r, n_r}_{p,q\,:\, p_1 + P_1, q_1 + Q_1 + 1 ; \ldots ; p_r + P_r, q_r + Q_r + 1}$$

$$\left[\begin{matrix} y_1 b_1^{w_1/t_1} \\ y_r b_r^{w_r/t_r} \end{matrix} \middle| \begin{matrix} U \\ V \end{matrix} \right],$$

where U stands for

$$(a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,p} : (c_j', \gamma_j')_{1,p_1}, \left(e_j' + \frac{\lambda_1 - \rho_1 + k_1}{t_1} E_j' ,\right.$$

$$\left. E_j' \frac{w_1}{t_1} \right)_{1,P_1} ; \ldots ; (c_j^{(r)}, \gamma_j^{(r)})_{1,P_r},$$

$$\left(e_j^{(r)} + \frac{\lambda_r - \rho_r + k_r}{t_r} E_j^{(r)}, \frac{w_r}{t_r} E_j^{(r)} \right)_{1,P_r}$$

and V stands for

$$(b_j ; \beta_j', \ldots, \beta_j^{(r)})_{1,q} : (d_j', \delta_j')_{1,m_1},$$

$$\left(f_j' + F_j' \frac{\lambda_1 - \rho_1 + k_1}{t_1}, F_j' \frac{w_1}{i_1} \right)_{1,Q_1}, (d_j', \delta_j')_{m_1+1,q_1},$$

$$(1 - \rho_1 + k_1, w_1) ; \ldots ; (d_j^{(r)}, \delta_j^{(r)})_{1,m_r},$$

$$\left(f_j^{(r)} + F_j^{(r)} \frac{\lambda_r - \rho_r + k_r}{t_r} , F_j^{(r)} \frac{w_r}{t_r}\right)_{1,Q_r} , \left(d_j^{(r)} , \delta_j^{(r)}\right)_{m_r+1, q_r} ,$$

$$(1 - \rho_r + k_r , w_r)$$

provided that $\{ Re \ (s_i), w_i , t_i \} > 0$,

$U^i > 0 , | \arg (b_i) | < \frac{1}{2} U^{(i)} \pi \ (i = 1, \ldots, r)$

$$Re(\lambda_i - \rho_i) + t_i \min_{1 \leqslant j \leqslant m_i} Re\left(f_j^{(i)} / F_j^{(i)}\right) - \omega_i$$

$$\max_{1 \leqslant j \leqslant m_i} Re \left\{ \left(c_j^{(i)} - 1 \right) / \gamma_j^{(i)} \right\} > 0$$

and the multiple series on the right-hand side is absolutely convergent.

Substituting these values of $g (x_1 , \ldots , x_r)$ and $h (s_1 , \ldots , s_r)$ from (4.2) and (4.4), respectively , in (2.2), altering the parameters of the H-function of several variables occuring on the right–hand side of (2.2) slightly, we arrive at the following interesting integral after a litile simplification .

$$(4.5) \quad \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ x_i^{\rho_i - 1} (x_i + s_i)^{-\lambda_i} H_{P_i,Q_i}^{M_i,0} \left[b_i (x_i + s_i)^{-t_i} \middle| \begin{matrix} (e_j^{(i)} , E_j^{(i)})_{1,P_i} \\ (f_j^{(i)} , F_j^{(i)})_{1,Q_i} \end{matrix} \right] \right. . H_1 \left[y_1 x_1^{\omega_1} , \ldots , y_r x_r^{\omega_r} \right] dx_1 \ldots dx_r$$

$$= \prod_{i=1}^{r} \left\{ \frac{b_i^{(\rho_i - \lambda_i)/t_i}}{t_i} \right\} \sum_{k_i=0}^{\infty} \frac{(-1)^{k_i} \left(s_i b_i^{-1/t_i} \right)^{k_i}}{k_i !}$$

$$. \ H_{p,q \,:\, p_1+P_1+1,\, q_1+Q_1+1 \,;\, \ldots \,;\, p_r+P_r+1 ,\, q_r+Q_r+1}^{0,0 \,;\, m_1+M_1,\, n_1+1 \,;\, \ldots \,;\, m_r+M_r ,\, n_r+1} \left[\begin{matrix} y_1 b_1^{w_1/t_1} \\ \vdots \\ y_r b_r^{w_r/t_r} \end{matrix} \middle| \begin{matrix} U' \\ \\ V' \end{matrix} \right]$$

where U' stands for

$$(a_j\,;\alpha_j'\,;\ \ldots,\alpha_j^{(r)})_{1,p}\ :\ (1-\rho_1, w_1),\ (c_j',\gamma_j')_{1,p_1},$$

$$(e_j' + \frac{\lambda_1-\rho_1+k_1}{t_1}E_j',\ \frac{w_1}{t_1}E_j)_{1,P_1};\ldots;(1-\rho_r, w_r),\ (c_j^{(r)},\gamma_j^{(r)})_{1,p_r},$$

$$(e_j^{(r)} + \frac{\lambda_r-\rho_r+k_r}{t_r}E_j^{(r)}\ ;\ \frac{w_r}{t_r}E_j^{(r)})_{1,P_r}$$

V' stands for

$$(b_j\,;\beta_j',\ldots,\beta_j^{(r)})_{1,q_1} : (d_j',\delta_j')_{1,m_1},$$

$$(f_j' + \frac{\lambda_1-\rho_1+k_1}{t_1}F_j'\ \frac{w_1}{t_1}F_j')_{1,Q_1},$$

$$(d_j',\delta_j')_{m_1+1,q_1},\ (1-\rho_1+k_1,w_1);\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,m_r},$$

$$(f_j^{(r)}+\frac{\lambda_r-\rho_r+k_r}{t_r}F_j^{(r)},\ \frac{w_r}{t_r}F_j^{(r)})_{1,Q_r},\ (d_j^{(r)},\delta_j^{(r)})_{m_r+1,q_r},$$

$$(1-\rho_r+k_r,w_r)\ .$$

$Re(\rho_i) > 0$, and the set of conditions mentined with (4.4) are satisfied .

The integral (4.5) is believed to be one of the most general integrals evaluated so far ; it isc apable of yielding a number of interesting integrals involving simpler functions of one , two or more variables as its particular cases .

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ON THE BESSEL TRANSFORMS

By

J. M. MÈNDEZ
Depártamento de Análisis Matemático, Facultad de Matemáticas,
Universidad de La Laguna, La Laguna (Tenerife),
Canary Islands, Spain

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ABSTRACT

In this paper, a new variant of the Hankel transform is introduced. The corresponding inversion theorem is established and some operational formulas are obtained. These, results are then used in solving certain partial differential equations. Finally, we show that this transform is connected with the Hankel-Bessel-Schwartz transform by means of an interesting Parseval type relation.

1. INTRODUCTION

A. L. Schwartz [8] first established the inversion theorem of the following modified Hankel transformation

$$F_1(y) = (h_{1,\mu} f)(y) = \int_0^\infty x^{2\mu+1} b_\mu(xy) f(x)\,dx , \tag{1}$$

where $b_\mu(z) = z^{-\mu} J_\mu(z)$, $J_\mu(z)$ being the Bessel function of the first kind of order μ. This transform has been investigated in distributional spaces by L. S. Dube and J. N. Pandey [2], W. Y. Lee [4], J. M. Méndez [6], and also by G. Altenburg [1] who calls it the Bessel transformation.

In this paper the following new variant of the Hankel transform

$$G_2(y) = (h_{2,\mu}\, g)(y) = y^{2\mu+1}\int_0^\infty b_\mu(xy)\, g(x)\, dx\,, \tag{2}$$

connected closely with (1), is considered and its inversion formula is given . Some important operational formulas are obtained, which will be used in solving a certain class of partial differential equations. Finally, we prove that both transforms (1) and (2) are related by means of Parseval's equations . These Parseval relations suggest new definitions of the transforms $h_{1,\mu}$ and $h_{2,\mu}$ on adequate spaces of generalized functions .

2. Preliminary results

The function $b_\mu(x)$ is a solution of the differential equation [1, p. 213]

$$y'' + (2\mu+1)\, x^{-1} y' + y = 0\,,$$

and, therefore, satisfies

$$x^2\, b_{\mu+2}(x) - 2(\mu+1)\, b_{\mu+1}(x) + b_\mu(x) = 0\,, \tag{3}$$

In this work we shall need the formulas [3, p. 16]

$$D_y\,[\, y^{2\mu}\, b_\mu(uy)\,] = y^{2\mu-1}\, b_{\mu-1}(uy) \tag{4}$$

$$D_y\, b_\mu(uy) = -\, yu^2\, b_{\mu+1}(uy) \tag{5}$$

An integration by parts and the use of (4) and (5) yield

$$\int_0^\lambda u^{2\mu+1}\, b_\mu(xu)\, b_\mu(yu)\, du =$$

$$= \frac{\lambda^{2\mu+2}}{x^2-y^2}\left[\, x^2\, b_{\mu+1}(\lambda x)\, b_\mu(\lambda y) - y^2 b_\mu(\lambda x)\, b_{\mu+1}(\lambda y)\,\right], \tag{6}$$

provided that $\mu \geqslant -\frac{1}{2}$.

From the asymptotic expansion of the Bessel functions [7, *p*. 122] we find that

$$b_{\mu}(x) = \sqrt{\left(\frac{2}{\pi}\right)}\, x^{-\mu-1/2} \cos\left(x - \frac{\mu\pi}{2} - \frac{\pi}{4}\right), \text{ as } x \to \infty \tag{7}$$

We also require the fact that [7, *p*. 23]

$$b_{\mu}(x) = O(1) \quad , \quad \text{as } x \to 0+ \tag{8}$$

Recall finally the value of the integral , $\mu > -1$ [10, *p*. 406]

$$\int_0^{\infty} t^{2\mu+1}\, b_{\mu+1}(at)\, b_{\mu}(ct)\, dt = \begin{cases} a^{-2\mu-2} & , a > c \\ \frac{1}{2}\, c^{-2\mu-2} & , a = c \\ 0 & , a < c \end{cases} \tag{9}$$

3. Inversion formula

THEOREM 1. *Let* $\mu \geqslant -\frac{1}{2}$. *If* $f(y)$ *is an integrable function such that*

$$\int_0^{\infty} y^{-\mu-1/2}\, f(y) \,|\, dy < \infty ,$$

then

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^{\lambda} u^{2\mu+1}\, b_{\mu}(xu) \int_0^{\infty} b_{\mu}(yu)\, f(y)\,dy\, du = \tfrac{1}{2}\,[\, f(x+0) + f(x-0)\,],$$

in a neighborhood of every point $y = x$, *where* $f(y)$ *is of bounded variation* .

Proof . The following proof, just as Schwartz's proof [8], is essentially that given by Titchmarsh [9] in his verification of the inversion theorem for the Hankel transform . Thus, suppose that x is fixed in $0 < x < \infty$. Through an argument similar to the one employed in [8], it can be deduced that

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^\lambda u^{2\mu+1}\, b_\mu(xu) \int_0^{x-\delta} b_\mu(yu)\, f(y)\, dy\, du = 0\,, \tag{10}$$

δ being a small positive number. However, we are interested in emphasizing the next result

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^\lambda u^{2\mu+1}\, b_\mu(xu) \int_{x+\delta}^\infty b_\mu(yu)\, f(y)\, dy\, du = 0\,. \tag{11}$$

Indeed, since $| z^{\mu+\frac{1}{2}}\, b_\mu(z) | = O(1)$ in $0 < z < \infty$, it follows that

$$| u^{\mu+\frac{1}{2}} \int_{x+\delta}^\infty b_\mu(yu)\, f(y)\, dy | = O\left(\int_0^\infty y^{-\mu-1/2}\, | f(y) |\, dy \right).$$

Then, we may invoke Fubini's theorem to obtain, in view of (6),

$$x^{2\mu+1} \int_0^\lambda u^{2\mu+1}\, b_\mu(xu) \int_{x+\delta}^\infty b_\mu(yu)\, f(y)\, dy\, du$$

$$= x^{2\mu+3}\, \lambda^{2\mu+2}\, b_{\mu+1}(\lambda x) \int_{x+\delta}^\infty b_\mu(\lambda y)\, (x^2-y^2)^{-1} f(y)\, dy \tag{12}$$

$$- x^{2\mu+1}\, \lambda^{2\mu+2}\, b_\mu(\lambda x) \int_{x+\delta}^\infty y^2 b_{\mu+1}(\lambda y)\, (x^2-y^2)^{-1} f(y)\, dy\,.$$

Note that the first term of the right-hand side of (12) can be written as

$$x^{\mu+\frac{3}{2}}\, (\lambda x)^{\mu+\frac{3}{2}}\, b_{\mu+1}(\lambda x) \int_{x+\delta}^\infty (\lambda y)^{\mu+1/2}\, b_\mu(\lambda y)\, (x^2-y^2)^{-1}\, y^{-\mu-1/}\, f(y)\, dy. \tag{13}$$

The factor outside of the integral sign is always bounded no matter what the value of λ may be, by virtue of (7) and (8). On the other hand, for very large values of λ, the integral in (13) reduces to

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_{x+\delta}^{\infty} \cos\left(\lambda y - \frac{\mu\pi}{2} - \frac{\pi}{4}\right)(x^2-y^2)^{-1}\, y^{-\mu-\frac{1}{2}} f(y)\, dy$$

which tends to zero as $\lambda \to \infty$, because of the Riemann – Lebesgue theorem. Hence, (13) vanishes when $\lambda \to \infty$. The same argument may be applied to the second term of (12). Thus we conclude that (11) holds.

Since $f(y)$ is of bounded variation in a neighborhood of $y=x$, we may put

$$y^{-2\mu-1} f(y) = x^{-2\mu-1} f(x+0) + \Psi_1(y) - \Psi_2(y),$$

on $(x, x+\delta)$, where Ψ_2 and Ψ_2 are positive and increasing functions such that $\Psi_1(x+0) = \Psi_2(x+0) = 0$. Then, by applying the second mean-value theorem, we arrive at

$$x^{2\mu+1}\int_0^{\lambda} b_\mu(xu)\, u^{2\mu+1}\int_x^{x+\delta} b_\mu(yu)\, f(y)\, dy\, du$$

$$= f(x+0)\int_0^{\lambda} b_\mu(xu\quad u^{2\mu+1}\int_x^{x+\delta} y^{2\mu+1} b_\mu(yu)\, dy\, du$$

$$+ \Psi_1(x+\delta)\int_0^{\lambda} u^{2\mu+1} b_\mu(xu)\int_{\xi}^{x+\delta} y^{2\mu+1} b_\mu(yu)\, dy\, du \tag{14}$$

$$- \Psi_2(x+\delta)\int_0^{\lambda} u^{2\mu+1} b_\mu(xu)\int_{\xi'}^{x+\delta} y^{2\mu+1} b_\mu(yu)\, dy\, du\ ,$$

with $x<\xi, \zeta'<x+\delta$. The first integral of the right-hand side of (14) tends to $\frac{1}{2} f(x+0)$ when $\lambda \to \infty$, as it follows from (4) and (9), whereas the remaining terms are $O(\varepsilon)$, since $\Psi_1(y) \leqslant \varepsilon$ and $\Psi_2(y) \leqslant \varepsilon$ for

all y , $x \leqslant y \leqslant x+\delta$, on choosing δ sufficiently small . Hence we have seen that

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^\lambda u^{2\mu+1} b_\mu(xu) \int_x^{x+\delta} b_\mu(yu) f(y)\, dy\, du = \tfrac{1}{2} f(x+0) \tag{15}$$

Next, we can derive in a similar way

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^\lambda u^{2\mu+1} b_\mu(xu) \int_{x-\delta}^{x} b_\mu(yu) f(y)\, dy\, du = \tfrac{1}{2} f(x-0) \tag{16}$$

The proof of the theorem is now completed cy combining (10), (11), (15) and (16) .

Remark 1 . Schwartz [8] proves the inversion theorem for the transform (1) by assuming that $x^{2\mu+1} f(x)$ is absolutely integrable on $0<x<\infty$ and that

$$\int_0^1 x^{\mu+1/2} \mid f(x) \mid dx$$

exists . These hypotheses can be simplified . Indeed, the absolute integrability of $x^{\mu+1/2} f(x)$ on $0<x<\infty$ is enough to guarantee the validity of the inversion formula for the transform (1). This point can be verified by proceeding as in the proof of Theorm 1 .

4. An operational calculus and its applications

In the sequel. we assume that $\mu \geqslant -\frac{1}{2}$. Let $f(x) \in C^1(0,\infty)$ such that $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to\infty} x^{-\mu-1/2} f(x) = 0$. Then, upon integrating by parts and making use of (5), we get

$$y^{2\mu+1} \int_0^\infty b_\mu(xy) f'(x)\, dx$$

$$= y^{2\mu+1} \left\{ \Big[b_\mu(xy)\ f(x) \Big]_{x\to 0}^{x\to\infty} + y^2 \int_0^\infty b_{\mu+1}(xy)\ x f(x)\ dx \right\}.$$

Hence

$$h_{2,\mu}\,[\,f'(x)\,] = h_{2,\mu+1}\,[\,f(x)\,] \tag{17}$$

If $f(x) \in C^2\,(0,\infty)$, and if $f^{(i)}(x)$ tends to *zero* as $x \to 0^+$, whereas $x^{-\mu-1/2} f^{(i)}(x)$ vanishes when $x \to \infty$ ($i=0, 1$), by using (17) and the formula for integrating by parts, we obtain

$$h_{2,\mu}\,[\,f''(x)\,] = h_{2,\mu+1}\,[\,xf'(x)\,]$$

$$= y^{2\mu+3} \int_0^\infty f(x)\ \{\, x^2 y^2\, b_{\mu+2}(xy) - b_{\mu+1}(xy)\,\}\, dx\ .$$

Finaliy, from (3) we have

$$h_{2,\mu}\,[\,f''(x)\,] = (2\mu+1)\,h_{2,\mu+1}\,[\,f(x)\,] - y^2 h_{2,\mu}\,[\,f(x)\,] \tag{18}$$

Now, in the above assumptions, it is deduced that

$$h_{2,\mu}\,[\,D^2 f(x) - (2\mu+1)\,D\,(x^{-1} f(x))\,] = -\,y^2\,h_{2,\mu}\,[\,f(x)\,]\,, \tag{19}$$

as a simple consequence of (17) and (18) .

This formula can be fruitfully utilized in solving a certain class of partial differential equations , We illustrate it with an example in which we find the solution $v = v\,(r,t)$ of the equation

$$\frac{\partial^2 u}{\partial r^2} - \frac{2\mu+1}{r}\frac{\partial v}{\partial r} + \frac{2\mu+1}{r^2}\,v + \frac{\partial^2 v}{\partial t^2} = 0\,,\quad \mu \geqslant 0\ , \tag{20}$$

satisfying the conditons

$v\,(r,t)$ and $\dfrac{\partial v\,(r;t)}{\partial r}$ vanish *as* $r \to 0^+$

$v\,(\,r,t\,) = O\,(\,r^{\mu+1/2})$ and $\dfrac{\partial v\,(\,r,t\,)}{\partial r} = O\,(\,r^{\mu+1/2})$, *as* $r \to \infty$

$v\,(\,r,t\,) \to 0$, *as* $t \to \infty$

$v\,(\,r,0\,) = f\,(\,r\,)$

Set $V\,(\,\rho,t\,) = h_{2,\mu}\,[\,u\,(\,r,t\,)\,]$. By applying $h_{2,\mu}$ to (20) and invoking formula (19), we convert (20) into

$$\frac{\partial^2 V}{\partial t^2} - \rho^2 V = 0 \quad , \quad 0 < \rho < \infty$$

$$V(\rho, t) \to 0 \ , \ as\ t \to \infty$$

$$V(\rho, 0) = F(\rho) = h_{2,\mu}[f(r)] \ ,$$

whose solution is

$$V(\rho, t) = F(\rho)\, e^{-\rho t}$$

Inverting this result , we obtain

$$v(r, t) = r^{2\mu+1} \int_0^\infty b_\mu(r\rho)\, F(\rho)\, e^{-\rho t}\, d\rho$$

as our formal solution of (20) .

5. Parseval relations for Bessel transforms

$L(0, \infty)$ denotes the space of all functions $f(x)$ that are Lebesgue integrable on $0 < x < \infty$, We now establish

THEOREM 2. *Let* $x^{\mu+1/2} f(x)$ *and* $y^{\mu+1/2} G_1(Y)$ *belong to* $L(0, \infty)$. *Assume that* $F(Y) = h_{1,\mu}[f(x)]$ *and* $g(x) = h_{1,\mu}^{-1}[G_1(y)] = h_{1,\mu}[G_1(Y)]$, *where* $\mu \geqslant -\frac{1}{2}$. *Then*

$$\int_0^\infty x^{2\mu+1} f(x)\, g(x)\, dx = \int_0^\infty y^{2\mu+1} F_1(Y)\, G_1(Y)\, dy \tag{21}$$

Proof . In fact, by the absolute convergence of the y-integral , we may invert the order of integration to obtain (21) .

Similarly, for the second Bessel transform . we have

THEOREM 3. *Suppose* $x^{-\mu-1/2} f(x)$ *and* $y^{-\mu-1/2} G_2(y)$ *belong to* $L(0, \infty)$, *where* $\mu \geqslant -\frac{1}{2}$. *If* $F_2(y) = h_{2,\mu}[f(x)]$ *and* $g(x) = h_{2,\mu}^{-1}[G_2(Y)] = h_{2,\mu}[G_2(Y)]$,

then

$$\int_0^\infty x^{-2\mu-1} f(x)\, g(x)\, dx = \int_0^\infty Y^{-2\mu-1} F_2(Y)\, G_2(Y)\, dy$$

We end this section by establishing a curious Parseval type relation, which involves both the Bessel transforms $h_{1,\mu}$ and $h_{2,\mu}$.

THEOREM 4 . *Let* $x^{\mu+1/2} f(x)$ *and* $Y^{-\mu-1/2} G_2(y)$ *belong to* $L(0,\infty)$, *where* $\mu \geqslant -\frac{1}{2}$, *If* $F_1(y) = h_{1,\mu}[f(x)]$ *and* $g(x) = \overset{-1}{h}_{2,\mu}[G_2(y)]$

$$= h_{2,\mu}[G_2(y)] ,$$

then $$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_1(y)\, G_2(y)\, dy ,$$

that is , $$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty h_{1,\mu}[f(x)] \,.\, h_{2,\mu}[g(x)]\, dy \tag{22}$$

Proof . The absolute convergence of the y-integral allows us to apply once more the Fubini's theorem to get

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty f(x) \left(x^{2\mu+1} \int_0^\infty b_\mu(x)\, G_2(y)\, dy \right) dx =$$

$$= \int_0^\infty G_2(y) \int_0^\infty x^{2\mu+1}\, b_\mu(xy)\, f(x)\, dx\, dy = \int_0^\infty F_1(y)\, G_2(y)\, dy \,.$$

Remark 2 . Note that the last Parseval relation does not involve any weight function, in contrast to the Parseval expressions of Theorems 2 and 3 . In fact, Zemanian's method for extending the Hankel transformation to a space of generalized functions of slow growth is based upon the existence of a Parseval relation analogous to (22) [11, *Th* . 5. 1–2, *p*. 127] Thus, the generalized Hankel transformation is defined as an extension of the mentioned Parseval equation [11,*p*. 142] . Hence, expression (22) suggests a new definition of the generalized Bessel transformations $h_{1,\mu}$ and $h_{2,\mu}$, quite different from those considered in [1], [2], [4], and [6] .

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HALL CURRENT EFFECT ON THERMAL INSTABILITY OF AN OLDROYDIAN VISCOELASTIC FLUID

By

R. C. SHARMA
Department of Mathematics, Himachal Pradesh University,
Shimla - 171 005, Himachal Pradesh, India

and

J. N. MISRA
Department of Mathematics, Government College,
Bilaspur - 174 001, Himachal Pradesh, India

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ABSTRACT

The thermal convection in a layer of electrically conducting Oldroydian viscoelastic fluid in the presence of Hall currents is studied . For the case of stationary convection, the Oldroydian fluid behaves like a Newtonian fluid and the magnetic field, Hall currents are found to have stabilizing and destabilizing effects respectively , The magnetic field and Hall currents introduce oscillatory modes in the system which were non-existent in their absence and when the strain retardation time be greater than the atress relaxation time .

1. INTRODUCTION

The problem of thermal convection in a horizontal layer of electrically conducting fluid in the presence of magnetic field has been discussed in detail by Chandrasekhar [3] . It is shown that a uniform magnetic field inhibits the onset of thermal convection . If an electric field is applied at right angles to the magnetic field, the whole current will not flow along the electric field . Hall current effect is this tendency of the electric current to flow across an electric field in the presence of a magnetic field . Gupta [5] has studied the problem of thermal convection in the presence of Hall current, The viscoelastic fluid described by the constitutive relations (1) (viz. eqs. (1) in section 2) is termed as an Oldroyd fluid ,

Eltayeb [4] has studied the convective instability in a rapidly rotating Oldroyd fluid . Sharma [7] has studied the stability of a layer of electrically conducting Oldroyd fluid (i. e. fluid obeying Oldroyd's constitutive equation) heated from below in the presence of magnetic field and has found the magnetic field to have a stabilizing influence , Sharma [8] has also studied the thermal instability of layer of Oldroyd fluid acted on by a uniform rotation and has found that the rotation has destabilizing as well as stabilizing effect under certain conditions in contrast to a Maxwell (viscoelastic) fluid where it has a destabilizing effect (Bhatia and Steiner [1]). An experimental demonstration by Toms and Strawbridge [9] reveals that a dilute solution of methyl methacrylate in *n*–butyl acetate agrees well with the theoretical model of Oldroyd fluid .

Keeping in mind the stability of some polymer solutions like a dilute solution of methyl methacrylate in *n*-butyl acetate, a re–consideration of the thermal convection in a layer of electrically conducting Oldroydian fluid in the presence of a uniform vertical magnetic field by including the Hall current is called for . This aspect forms the subject matter of the present paper .

2. Perturbation Equations

Consider an infinite horizontal layer of an electrically conducting Oldroydian fluid of depth d which is heated uniformly from below, acted on by a vertical magnetic field **H** (0, 0, H) and gravity force g (0, 0,$-g$) . The fluid is described by the constitutive relations

$$T_{ij} = - p\delta_{ij} + \tau_{ij} ;$$

$$\left(1+\lambda \frac{d}{dt}\right) \tau_{ij} = 2\mu \left(1+\lambda_0 \frac{d}{dt}\right) e_{ij} , \tag{1}$$

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

where T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, μ, λ and λ_0 denote respectively the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, viscosity, stress relaxation time and strain retardation time . v_i, x_i, p and d/dt are velocity vector, position vector, isotropic pressure and mobile operator respectively . Relations of the type (1) were first proposed by Jeffreys for Earth and studied by Oldroyd [6]. Oldroyd [6] slso showed that many rheological equations of state, of general validity, reduce to (1) when linearized .

Let v (u, v, w) , ρ, T, μ_e, N and e stand respectively for velocity, density, temperature, magnetic permeability, number density and charge of an electron; ν, η, χ and α denote the kinematic viscosity, the electrical resistivity, the thermal diffusivity and the coefficient of thermal expansion respectively . The suffix zero refers to values at the reference level $z=0$. Then the equations of momentum, mass, energy and the Maxwell's equations are

$$\left(1+\lambda \frac{d}{dt}\right)\left[\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}.\nabla\right)\mathbf{v}\right] = \left(1+\lambda \frac{d}{dt}\right)\left[-\frac{1}{\rho_0}\nabla p + \right.$$

$$\left(1+\frac{\delta\rho}{\rho_0}\right)\mathbf{g}+\frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\mathbf{H}\right)\times\mathbf{H}\Bigg]+\nu\left(1+\lambda_0\frac{d}{dt}\right)\nabla^2\mathbf{v}\ , \quad (2)$$

$$\nabla\cdot\mathbf{v}=0\ , \quad (3)$$

$$\frac{\partial T}{\partial t}+\left(\mathbf{v}\cdot\nabla\right)T=\chi\nabla^2 T\ , \quad (4)$$

$$\frac{\partial\mathbf{H}}{\partial t}=\left(\mathbf{H}\,\nabla\right)\mathbf{v}+\eta\nabla^2\mathbf{H}-\left(\frac{c}{4\pi Ne}\right)\nabla\times\left[\left(\nabla\times\mathbf{H}\right)\times\mathbf{H}\right], \quad (5)$$

$$\nabla\cdot\mathbf{H}=0, \quad (6)$$

where

$$\rho=\rho_0\left[1-\alpha\left(T-T_0\right)\right]\ . \quad (7)$$

In writing eq, (2), use has been made of the Boussinesq approximation . (7) represents the equation of state . The steady state solution is

$$\mathbf{v}=0\ ,\ T=T_0-\beta z\ ,\ \rho=\rho_0\left(1+\alpha\beta z\right)\ , \quad (8)$$

where $\beta\left(=\left|\frac{dT}{dz}\right|=\frac{T_0-T_1}{d}\right)$ is the magnitude of uniform adverse temperature gradient and is positive , The change in density $\delta\rho$ caused by the perturbation θ in temperature is given by

$$\delta\rho=-\alpha\rho_0\,\theta\ . \quad (9)$$

On linearization, eqs. (2)—(6) give

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial \mathbf{v}}{\partial t}=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\frac{1}{\rho_0}\nabla\delta p+\frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\mathbf{h}\right)\times\mathbf{H}-g\alpha\theta\right]+\nu\left(1+\lambda_0\frac{\partial}{\partial t}\right)\nabla^2\mathbf{v}\ , \tag{10}$$

$$\nabla\cdot\mathbf{v}=0\ ,\quad \nabla\cdot\mathbf{h}=0, \tag{11}$$

$$\frac{\partial\theta}{\partial t}=\beta w+\chi\nabla^2\theta\ , \tag{12}$$

$$\frac{\partial\mathbf{h}}{\partial t}=\left(\mathbf{H}\cdot\nabla\right)\mathbf{v}+\eta\nabla^2\mathbf{h}-\left(\frac{c}{4\pi Ne}\right)\nabla\times\left[\left(\nabla\times\mathbf{h}\right)\times\mathbf{H}\right]. \tag{13}$$

Equations (10) – (13) give

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{\partial}{\partial t}\nabla^2 w-g\alpha\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)-\frac{\mu_e H}{4\pi\rho_0}\frac{\partial}{\partial z}\nabla^2 h_z\right]$$

$$=\nu\left(1+\lambda_0\frac{\partial}{dt}\right)\nabla^4 w\ , \tag{14}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\zeta}{\partial t}-\nu\left(1+\lambda_0\frac{\partial}{\partial t}\right)\nabla^2\zeta=\frac{\mu_e H}{4\pi\rho_0}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\xi}{\partial z}\ , \tag{15}$$

$$\left(\frac{\partial}{\partial t}-\chi\nabla^2\right)\theta=\beta w\ , \tag{16}$$

$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)\xi=H\frac{\partial\zeta}{\partial z}+\frac{cH}{4\pi Ne}\nabla^2\frac{\partial hz}{\partial z}\ , \tag{17}$$

$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)h_z=H\frac{\partial w}{\partial z}-\frac{cH}{4\pi Ne}\frac{\partial\xi}{\partial z}\ . \tag{18}$$

Here the boundaries are taken to be free as well as perfect conductors of heat and the adjoining medium is taken to be electrically nonconducting. Though the case of two free boundaries is artificial except in the case of stellar atmospheres, it allows us to obtain the analytical solution without affecting the essential features of the problem. The boundary conditions in this case are

$$w = \frac{\partial w^2}{\partial z^2} = \theta = 0 \, , \; \frac{\partial \zeta}{\partial z} = 0 \, , \; \xi = 0 \quad \text{at } z = 0 \text{ and } z = d \, , \qquad (19)$$

and **h** are continuous with an external vacuum field on a nonconducting boundary.

3. Dispersion Relation

We now analyze the disturbances into normal modes, assume that the perturbations w, θ, h_z, ζ and ξ have the forms

$$[\, w, \theta, h_z, \zeta, \xi \,] = [\, W(z), \otimes(z), K(z), Z(z), X(z) \,] \; exp\,(ik_x x + ik_y y + nt) \, , \qquad (20)$$

where n is the growth rate, $k = (\, k_x^2 + k_y^2 \,)^{\frac{1}{2}}$ and k_x, k_y are wave numbers along the x and y directions respectively.

Putting the coordinates x, y, z in the new unit of length d and letting $a = kd$, $\sigma = nd^2/\nu$, $F = \lambda\nu/d^2$, $\Gamma = \lambda_0\nu/d^2$, $p_1 = \nu/\chi$, $p_2 = \nu/\eta$ and $D = d/dz$, the non-dimensional forms of eqs. (14)-(18), using expression (20), are

$$(\, 1 + F\sigma \,) \left[\sigma \,(\, D^2 - a^2 \,)\, W + \left(\frac{g\alpha d^2}{\nu} \right) a^2 \otimes - \frac{\mu_e H d}{4\pi\rho_0\nu} (D^2 - a^2)\, DK \right]$$

$$= (\, 1 + \Gamma\sigma \,)\, (D^2 - a^2\,)^2\, W \; , \qquad (21)$$

$$\left[[1+F\sigma) \sigma - (1+\Gamma\sigma) (D^2 - a^2) \right] Z = \frac{\mu_e Hd}{4\pi\rho_0\nu} (1+F\sigma) DX , \tag{22}$$

$$(D^2-a^2-p_1\sigma) \otimes = - \left(\frac{\beta d^2}{\chi} \right) W , \tag{23}$$

$$(D^2-a^2-p_2\sigma) X = - \left(\frac{Hd}{\eta} \right) DZ - \frac{cH}{4\pi Ne\eta d} (D^2-a^2) DK , \tag{24}$$

$$(D^2-a^2-p_2\sigma) K = - \left(\frac{Hd}{\eta} \right) DW + \left(\frac{cHd}{2\pi Ne\eta} \right) DX , \tag{25}$$

where $D = d/dz$. Eliminating Z, X, K and $\otimes$ from eqs . (21) — (25), we obtain

$$[\{\sigma(1+F\sigma)-(1+\Gamma\sigma) (D^2-a^2)\} \{(D^2-a^2-p_2\sigma)^2 + MD^2 (D^2-a^2)\}$$
$$+ QD^2 (D^2-a^2-p_2\sigma) (1+F\sigma)] [(D^2-a^2) (D^2-a^2-p_1\sigma)$$
$$- (1+ F\sigma) Ra^2] W + Q (1+F\sigma) (D^2-a^2) (D^2-a^2-p_1\sigma)$$
$$[\{(1+F\sigma) \sigma - (1+\Gamma\sigma) (D^2-a^2)\} (D^2-a^2-p_2\sigma)$$
$$+ QD^2 (1+F) D^2 W = 0 , \tag{26}$$

where $Q = \frac{\mu_e H^2 d^2}{4\pi\rho\nu\eta}$ is the Chandrasekhar number , $R = \frac{g\alpha\beta d^4}{\nu\chi}$ is the Rayleigh number and $M = \left(\frac{cH}{4\pi Ne\eta} \right)^2$ is a non-dimensional number accounting for the Hall effects .

The boundary conditions (19) transform to

$$\left.\begin{array}{l} W = D^2W = \otimes = DZ = X = 0 \text{ at } z = 0 \text{ and } z = 1 \\ \text{and } \mathbf{h} \text{ is continuous with an external vaccum field .} \end{array}\right\} \tag{27}$$

Using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish for $z = 0$ and 1 and hence the proper solution of (26) characterizing the lowest mode is

$$W = W_0 \sin \pi z \, , \tag{28}$$

where W_0 is a constant . Substituting (28) in eq . (26) , we obtain the dispersion relation

$$R_1 x = \frac{(1+x)(1+x+ip_1\sigma_1)}{(1+i\sigma_1\pi^2 F)} \left[(1+x)(1+i\sigma_1\pi^2\Gamma) + i\sigma_1 (1+i\sigma_1\pi^2 F) \right]$$

$$+ \frac{Q_1(1+x)(1+x+ip_1\sigma_1)\,[\{(1+x)(1+i\sigma_1\pi^2\Gamma) + i\sigma_1 (1+i\sigma_1\pi^2 F)\}}{[\{(1+x)(1+i\sigma_1\pi^2\Gamma) + i\sigma_1 (1+i\sigma_1\pi^2 F)\}}$$

$$\times \frac{(1+x+ip_2\sigma_1) + Q_1 (1+i\sigma_1\pi^2 F)]}{\{(1+x+ip_2\sigma_1)^2 + M(1+x)\} + Q_1 (1+i\sigma_1\pi^2 F)(1+x+ip_2\sigma_1)]} \tag{29}$$

where $x = a^2 / \pi^2$, $R_1 = R/\pi^4$, $Q_1 = Q/\pi^2$ and $i\sigma_1 = \sigma/\pi^2$.

In the limit of vanishing M , the dispersion relation (29) reduces to the result (Sharma [7]) . In the limit of vanishing M and Γ , the dispersin relation also reduces to the case of a Maxwell fluid (Bhatia and Steiner [2]) .

4. The Stationary Convection

For the case of stationary convection *i. e.* $\sigma = 0$, eq . (29) reduces to

$$R_1 = \left(\frac{1+x}{x}\right)\left[(1+x)^2 + \frac{Q_1\{(1+x)^2 + Q_1\}}{(1+x)(1+x+M)+Q_1}\right], \tag{30}$$

the result obtained by Gupta [5].

We thus find that for the stationary convection, the stress relaxation time parameter F and the strain retardation time parameter Γ vanish with σ and the Oldroydian viscoelastic fluid behaves like a Newtonian viscous fluid. To find the effect of magnetic field and Hall current on the system, we examine the natures of dR_1/dQ_1 and dR_1/dM analytically.

From eq. (30), it follows that

$$\frac{dR_1}{dQ_1} = \frac{(1+x)^2}{x}\cdot\frac{\{(1+x)^2 + 2Q_1\}(1+x+M) + Q_1}{[(1+x)(1+x+M)+Q_1]^2}, \tag{31}$$

which is positive and so the magnetic field has a stabilizing effect for stationary convection.

Equation (30) also yields

$$\frac{dR_1}{dM} = -\frac{Q_1(1+x)^2\{(1+x)^2 + Q_1\}}{x[(1+x)(1+x+M)+Q_1]^2}, \tag{32}$$

which is always negative. The Hall current therefore has a destabilizing effect on the system for stationary convection.

5. Stability of the System and Oscillatory Modes

Multiplying eq. (21) by W^*, the complex conjugate of W, integrating over the range of z and using (22)-(25), we obtain

$$\sigma I_1 + \frac{(1+\Gamma\sigma)}{(1+F\sigma)} I_2 - \frac{g\alpha\chi a^2}{\nu\beta}(I_3 + \sigma^* p_1 I_4) + \frac{\mu_e\eta}{4\pi\rho\nu}(I_5 + \sigma^* p_2 I_6)$$

$$+ \frac{\mu_e \eta d^2}{4\pi\rho\nu} (I_7 + \sigma p_2 I_8) + d^2 [\sigma^* I_9 + \frac{1+\Gamma\sigma^*}{1+F\sigma^*} I_{10}] = 0 \ , \tag{33}$$

and $I_1 - I_{10}$ are all positive definite and are given in the appendix , Putting $\sigma = \sigma_r + i \sigma_i$ in eq . (33) and separating the real and imaginary parts , we obtain

$$\sigma_r \left[I_1 + \frac{(F+\Gamma) (I_2 + d^2 I_{10})}{(1+\sigma_r F)^2 + \sigma_i^2 F^2} + \frac{\mu_e \eta}{4\pi\rho\nu} p_2 (I_6 + d^2 I_8) + d^2 I_9 - \frac{g\alpha\chi a^2}{\nu\beta} p_1 I_4 \right]$$

$$= - \left[\frac{(I_2 + d^2 I_{10}) \{ 1 + F\Gamma (\sigma_r^2 + \sigma_i^2) \}}{(1+\sigma_r F)^2 + \sigma_i^2 F^2} + \frac{\mu_e \eta}{4\pi\rho\nu} (I_5 + d^2 I_7) - \frac{g\alpha\chi a^2}{\nu\beta} I_3 \right] , \tag{34}$$

and

$$\sigma_i \left[I_1 + \frac{(\Gamma - F) (I_2 - d^2 I_{10})}{(1+\sigma_\gamma F)^2 + \sigma_i^2 F^2} + \frac{\mu_e \eta d^2}{4\pi\rho\nu} b_2 I_8 + \frac{g\alpha\chi a^2}{\nu\beta} p_1 I_4 - \frac{\mu_e \eta}{4\pi\rho\nu} p_2 I_6 - d^2 I_9 \right] = 0 \ . \tag{35}$$

It is inferred from eq (34) that σ_r may be positive or negative meaning thereby that the systèm may be stable or unstable . Equation (35) implies that in the absence of magnetic field (and hence Hall currents), the oscillatory modes are not allowed if the strain retardation time λ_0 is greater than the stress relaxation time λ $\left[\text{for } \Gamma \left(= \frac{\lambda_0 \nu}{d^2}\right) > F \left(= \frac{\lambda \nu}{d^2}\right)\right]$. The magnetic field and Hall currents, thus, introduce oscillatory modes in the system which were non-existent in their absence and when the strain retardation time be greater than the stress relaxation time.

6, Results

The conclusions of the paper are presented here.

(i) The Oldroyd viscoelastic fluid behaves like a Newtonian fluid for the case of stationary convection. For stationaro convection, the magnetic field and Hall currents are found to have stabilizing and destabilizing effects respectively.

(ii) The magnetic field introduces oscillatory modes in the system which were non-existent in its absence and when the strain retardation time be greater than the stress relaxation time.

(iii) Hall currents introduce oscillatory modes in the system which were not present in the absence of Hall currents and when the strain retardation time be greater than the stress relaxation time.

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Appendix

The integrals $I_1 - I_{10}$ in eq. (33) are given as under :

$$I_1 = \int_0^1 (| DW |^2 + a^2 | W |^2) \, dz \,,$$

$$I_2 = \int_0^1 (| D^2W |^2 + 2a^2 | DW |^2 + a^4 | W |^2) \, dz \,,$$

$$I_3 = \int_0^1 (| D\otimes |^2 + a^2 | \otimes |^2) \, dz \,,$$

$$I_4 = \int_0^1 | \otimes |^2 \, dz \,,$$

$$I_5 = \int_0^1 (| D^2K |^2 + 2 a^2 | DK |^2 + a^4 | K |^2) \, dz \,,$$

$$I_6 = \int_0^1 (| DK |^2 + a^2 | K |^2) \, dz \,,$$

$$I_7 = \int_0^1 (| DX |^2 + a^2 | X |^2) \, dz \,,$$

$$I_8 = \int_0^1 | X |^2 \, dz \,,$$

$$I_9 = \int_0^1 | z |^2 \, dz \,,$$

$$I_{10} = \int_0^1 (| Dz |^2 + a^2 | z |^2) \, dz \,.$$

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A RESULT IN THE GEOMETRY OF PROBABILITY SPACE

By

J. N. KAPUR*

Department of Mathematics, Iadian Institute of Technology,

Kanpur - 208016, India

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ABSTRACT

It ts shown that the set of those probability distributions p from which the weighted average of directed divergences to k given probability distributions $Q_1, Q_2, \ldots, Q_k$ is constant is also the set of distributions P from which the directed divergence to a fixed distribution Q is constant. In other words, it is shown that, in probability space, every 'ellipse' is a 'circle'.

1. BASIC DEFINITIONS

A *probability space* is the set of all probability distributions

$$P = (p_1 , p_2 , \ldots , p_n) \tag{1}$$

where

$$p_1 \geqslant 0 , p_2 \geqslant 0 , \ldots , p_n \geqslant 0 , \sum_{i=1}^{n} p_i = 1 . \tag{2}$$

Every probability distribution is represented by a point in ihis space. The point P is called a *non-degenerate point*, if none of the probabilities is zero; otherwise it is called a *degenerate point*.

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*Current address : Department of mathematics,
Institute of Technology, Hauz Khas, New Delhi-110016, India.

For different values of n, we get different probability spaces. A degenerate point in a higher dimensional space may be regarded as a non-degenerate point in a lower dimensional space. In our discussion, we shall consider a specified probability space *i.e*, we shall consider n to be fixed.

The *distance* of a point P from a point Q is measured by the Kullback-Leibler measure (see [8]) :

$$D (P : Q) = \sum_{i=1}^{n} p_i \ln (p_i/q_i) . \tag{3}$$

It can easily be shown [7] that $D (P : Q) \geqslant 0$ and $D (P : Q) = 0$ iff $P = Q$. It is also obvious that this distance is not symmetric since the distance of Q from P viz. $D (Q : P)$ is not necessarily equal to $D (P : Q)$, the distance of P from Q. It is also obvious that $D (P : Q)$ is finite only if either Q is non-degenerate or if Q is degenerate then P is also degenerate and corresponding to every zero component of Q, the corresponding component of P is also zero.

The posnts P corresponding to all those distributions P for which $D (P : Q) = K$ are said to lie on a *circle of the first type* with centre Q and radius K. In the same way the points corres-ponding to all those distributions P for which $D (Q : P) = K$ are said to lie on a *circle of the second type* with centre Q and radius K. These two circles are in general quite different.

The points P corresponding to all those distributions P for which $\lambda_1 D (P : Q_1) + \lambda_2 D (P : Q_2) + \ldots + \lambda_k D (P : Q_k) = K$, where Q_1, Q_2, …, Q_k are fixed probability distributions and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are fixed non-negative constants whose sum is unity and K is a constant, are said to lie on a *weighted k-ellipse of the first type* with focii at

$Q_1 , Q_2 , \ldots , Q_k$. It will be called a *weighted k-elliptic of the second type if* $\lambda_1 D(Q_1 : P) + \lambda_2 D(Q_2 : P) + \cdots + \lambda_k D(Q_k : P) = K$.

If $\lambda_1 = \lambda_2 = \ldots = \lambda_k$, these will be called *k-ellipses of the first and second type* . If $k = 2$, these will be called *ellipses of the first and second type* . If $k = 1$, these reduce to *circles of the first and second type* .

2. CIRCLES AND ELLIPSES OF THE FIRST TYPE

(a) *when Q is non-degenerate*, the circle of the first type is given by

$$\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} = K . \tag{4}$$

Since for given non-zero $q_1, q_2 , \ldots , q_n$, $\sum_{i=1}^{n} \ln (p_i/q_i)$ is a convex function of $p_1 , p_2 , \ldots , p_n$, its maximum value occurs at one of the vertices of the simplex (*2*) and

$$\max_{p} \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i} = -\ln g_{min} \tag{5}$$

where

$$g_{min} = min (q_1 , q_2 , \ldots , q_n) \tag{6}$$

Thus if $k > -\ln g_{min}$, there is no probability distributions P satisfying (4) and as such the circle is imaginary . If $K = -\ln g_{min}$, there is just one probability distribution P satisfying (4), If $K = 0$, there is again one probability distribution viz . Q satisfying (4) . When K lies between 0 and $-\ln g_{min}$, there may be an infinity of probability distributions satisfying (4) and as such there may be an infinity of points on the circle with centre Q and radius K .

(b) When $Q_1 , Q_2 , \ldots , Q_k$ are non-degenerate, the weighted

k-ellipses of the first type with focii at $Q_1, Q_2, \ldots, Q_k$ is given by

$$\lambda_1 D(P : Q_1) + \lambda_2 D(P : Q_2) + \ldots + \lambda_k D(P : Q_k) = K$$

or $$\sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_{ij}} = K \tag{7}$$

or $$\sum_{i=1}^{n} p_i \ln \frac{p_i}{\hat{q}_i} = K, \tag{8}$$

where

$$\hat{q}_i = \prod_{j=1}^{k} q_{ij}^{\lambda_j} \tag{9}$$

is the weighted geometrical mean of the i th components of $Q_1, Q_2, \ldots, Q_k$. Let

$$A = \sum_{i=1}^{n} \hat{q}_i . \tag{10}$$

From (8) and (10)

$$\sum_{i=1}^{n} p_i \ln \frac{p_i}{\hat{q}_i / A} = \sum_{i=1}^{k} p_i \ln A + K$$

or $$D(P : \tilde{Q}) = K' \tag{11}$$

where

$$\tilde{Q} = \left(\frac{\hat{q}_1}{A}, \frac{\hat{q}_2}{A}, \ldots, \frac{\hat{q}_n}{A} \right) \tag{12}$$

is a certain probability distribution whose i th component is proportional to the weighted geometrical mean of the ith components of $Q_1, Q_2, \ldots, Q_k$ and

$$K' = K + \ln A \tag{13}$$

is a new constant.

Now since the weighted geometric mean of k positive numbers $\leqslant$ their weighted arithmetic mean, we get from (9) and (10)

$$\hat{q}_i = \sum_{j=1}^{k} q_{ij}^{\lambda_j} \leqslant \sum_{j=1}^{k} \lambda_j q_{ij} ,$$

so that $A = \sum_{i=1}^{n} \hat{q}_i \leqslant \sum_{i=1}^{n} \sum_{i=1}^{k} \lambda_j q_{ij} = \sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} q_{ij} = \sum_{j=1}^{k} \lambda_j = 1,$

$$(14)$$

giving $A \leqslant 1$, $\ln A \leqslant 0$, $K' \leqslant K$. (15)

Now (11) represents a circle with centre $\widetilde{Q}$ and radius K' . *Thus the weighted k–ellipse is a circle* . If the weights are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and the focii of the k–ellipse are $Q_1, Q_2, \ldots, Q_k$, then the centre of the circle is $\widetilde{Q}$ where $\widetilde{Q}$ is defined by (9), (10), (12). The circle will be real only if

$$0 \leqslant K + \ln A \leqslant -\ln \frac{\min(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n)}{A} \, . \tag{16}$$

If all the focii coincide, they will coincide with the centre of the circle.

If $K = -\ln A$, the radius of the circle is zero and thus $-\ln A$ gives the minimum value of $\lambda_1 D(P : Q_1) + \lambda_2 D(P : Q_2) + \ldots + \lambda_k D(P : Q_k)$ for variations of P so that

$$\min_P [\lambda_1 D(P : Q_1) + \lambda_2 D(P : Q_2) + \ldots + \lambda_k D(P : Q_k)] = -\ln A \tag{17}$$

Also $\max\limits_P [\lambda_1 D(P : Q_1) + \lambda_2 D(P : Q_2) + \ldots + \lambda_k D(P : Q_k)]$

$$= \max_P [K] = \max_P [K' - \ln A]$$

$$= -\ln \frac{\min(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n)}{A} - \ln A$$

$$= -\ln \min (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n) \tag{18}$$

so that $-\ln \sum_{i=1}^{n} q_i \leqslant \lambda_1 D(P : Q_1) + \lambda_2 D(P : Q_2) + \cdots + \lambda_k D(P : Q_k)$

$$\leqslant -\ln \min (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n) . \tag{19}$$

It may be noted that in Euclidean space, the weighted average of the distance if a point from k given fixed points has a minimum value, but no simple explicit expression is available for this and its maximum value is infinity .

(c) *When* Q *is degenerate* . If some components of Q are zero, then the corresponding components of P have also to be zero if $D(P : Q)$ is to be finite . If m components of Q are zero, then the same m components of P have to be zero and both P and Q belong essentially to $n-m$ dimensional probability space . The problem is thus reduced to a problem of type (a) in a lower dimensional space .

(d) *When some or all of* $Q_1, Q_2, \ldots, Q_k$ *are degenerate* . If the same components of $Q_1, Q_2, \ldots, Q_k$ are zero, the problem reduces to a problem in a lower dimensional space . If different components are zero, then all componennts of P corresponding to every zero components of $Q_1, Q_2, \ldots, Q_k$ have to be zero and P is highly restricted. In this case, it is better to use the second type of weighted ellipses .

3. CIRCLES AND ELLIPSES OF THE SECOND TYPE

(a) The distribution P belongs to the circle of the second type with centre Q and radius K if

$$\sum_{i=1}^{n} b_i \ln \frac{q_i}{p_i} = K . \tag{20}$$

If any component $q_i =$ zero, then we need not take the corresponding component p_i to be zero, but if q_i is not zero, then p_i cannot be zero since k is finite .

(b) The weighted ellipse in this case is given by

$$\sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} q_{ij} \ln \frac{q_{ij}}{p_i} = K \tag{21}$$

$$\text{or } -\sum_{i=1}^{n} \left(\sum_{j=1}^{k} \lambda_j q_{ij} \right) \ln p_i = K - \sum_{j=1}^{k} \sum_{i=1}^{n} \lambda_j q_{ij} \ln q_{ij}$$

$$\text{or } \quad -\sum_{i=1}^{n} \bar{q}_i \ln p_i = K - \sum_{j=1}^{k} \sum_{i=1}^{n} \lambda_j q_{ij} \ln q_{ij} \, . \tag{22}$$

This can be compared with the equation of the circle

$$-\sum_{i=1}^{n} q_i \ln p_i = K - \sum_{i=1}^{n} q_i \ln q_i \, . \tag{23}$$

$$\text{Now } \sum_{i=1}^{n} \bar{q}_i = \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_j q_{ij} = \sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} q_{ij} = \sum_{j=1}^{k} \lambda_j = 1, \tag{24}$$

$$\text{so that} \quad \bar{Q} = (\bar{q}_1 , \bar{q}_2 , \ldots , \bar{q}_n) \tag{25}$$

is a proper probability distribution . In fact

$$\bar{Q} = \lambda_1 Q_1 + \lambda_2 Q_2 + \ldots + \lambda_k Q_k \, , \tag{26}$$

is the weighted arithmetic mean of $Q_1 , Q_2 , \ldots , Q_k$. From (22) and (23) we find that the weighted k-ellipse of the second type with focii $Q_1 , Q_2 , \ldots , Q_k$ is a circle of the second type with centre $\bar{Q}$. Its radius is given by

$$R = \sum_{i=1}^{n} \bar{q}_i \ln \bar{q}_i + k - \sum_{j=1}^{k} \sum_{i=1}^{n} \lambda_j q_{ij} \ln q_{ij} \tag{27}$$

$$= -H(\bar{Q}) + K + \sum_{j=1}^{k} \lambda_j H(Q_j), \tag{28}$$

where $$H(P) = -\sum_{i=1}^{n} p_i \ln p_i \tag{29}$$

is Shannon's [10] measure of entropy so that

$$R = K - [\, H(\lambda_1 Q_1 + \lambda_2 Q_2 + \dots + \lambda_k Q_k) - \lambda_1 H(Q_1) - \dots - \lambda_k H(Q_k)\,] \tag{30}$$

$$= K - J(\lambda_1, \lambda_2, \dots, \lambda_k ; Q_1, Q_2, \dots, Q_k), \tag{31}$$

where $J(\lambda_1, \lambda_2, \dots, \lambda_k ; Q_1, Q_2, \dots, Q_k)$ is Jensen's difference [1] . Since $H(P)$ is a concave function of $p_1, p_2, \dots, p_n$, this Jensen's difference $\geqslant 0$ and vanishes iff $Q_1 = Q_2 = \dots = Q_k$.

Thus the weighted ellipse is a circle whose centre is the weighted arithmetic mean of $Q_1, Q_2, \dots, Q_k$ and whose radius $\leqslant K$ and is equal to K iff $Q_1 = Q_2 = \dots = Q_k$ *i.e.* iff all the focii coincide. The circle will be real iff

$$K \geqslant J(\lambda_1, \lambda_2, \dots, \lambda_k, Q_1, Q_2, \dots, Q_k) \tag{32}$$

If $K = J(\lambda_1, \lambda_2, \dots, \lambda_k ; Q_1, Q_2, \dots, Q_k)$, the circle become a point circle consisting of only one point viz $\bar{Q}$.

4. COMPARISON OF THE TWO TYPES OF WEIGHTED ELLIPSES

For both types, the weighted ellipse is a circle . The centres in the two cases are given by

$$\widetilde{Q} = \left(\frac{\hat{q}_1}{A}, \frac{\hat{q}_2}{A}, \dots, \frac{\hat{q}_n}{A} \right) \tag{33}$$

$$\tilde{Q} = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n) \tag{34}$$

The first centre is based on the weighted geometrical means of the ith components of $Q_1, Q_2, \ldots, Q_k$ and the second centre is based in their weighted arithmetic means. The radii of the two circles are

$$R_1 = K + \ln \left(\sum_{i=1}^{n} \hat{q}_i \right) \tag{35}$$

and $R_2 = K - [\, H(\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k) - \sum_{j=1}^{k} \lambda_j H(P_j) \,].$ (36)

The radii in both cases are less than K. In the first case the difference is the logarithm of the sum of the geometrical means and in the second case it is the Jensen's difference.

In the first case, there is an upper bound on K, in the second case there is none.

5. CONTINUOUS-VARIATE DISTRIBUTIONS CASE

(a) Let $P, Q, Q_1, Q_2, \ldots, Q_k$ distributions have density functions $f(x), g(x)$ $g_1(x), g_2(x), \ldots, g_k(x)$ respectively, then the first type of weighted ellipse is given by

$$\sum_{j=1}^{k} \lambda_j \int f(x) \ln \frac{f(x)}{g_j(x)} \, dx = K \tag{37}$$

or
$$\int f(x) \ln \frac{f(x)}{\hat{g}(x)} \, dx = K; \quad \hat{g}(x) = \prod_{j=1}^{k} g_j^{\lambda_j}(x) \tag{38}$$

or
$$\int f(x) \ln \frac{f(x)}{\hat{g}(x)/A} = K + \ln A; \; A = \int \hat{g}(x) \, dx \tag{39}$$

which is circle with centre of probability distribution with density function $\hat{g}(x)/A$ and with radius $K + \ln A$.

Since the geometric mean of a number of positive numbers $\leqslant$ the arithmetic mean

$$\hat{g}(x) \leqslant \sum_{j=1}^{k} \lambda_j g_j(x), \tag{40}$$

so that

$$A \leqslant \int \sum_{j=1}^{k} \lambda_j g_j(x)\, dx = \sum_{j=1}^{k} \lambda_j \int g_j(x)\, dx = \sum_{j=1}^{k} \lambda_j = 1 \tag{41}$$

and

$$\ln A \leqslant 0 . \tag{42}$$

(b) The weighted ellipse of the second type is given by

$$\sum_{j=1}^{k} \lambda_j \int g_j(x) \ln \frac{g_j(x)}{f(x)}\, dx = K \tag{43}$$

or

$$- \int \bar{g}(x) \ln f(x)\, dx \int K - \sum_{j=1}^{k} \lambda_j \int g_j(x) \ln g_j(x)\, dx \tag{44}$$

where

$$\bar{g}(x) = \sum_{j=1}^{k} \lambda_j g_j(x), \tag{45}$$

so that

$$\int \bar{g}(x) \ln \frac{\bar{g}(x)}{f(x)}\, dx = \int \bar{g}(x) \ln \bar{g}(x)\, dx + K - \sum_{j=1}^{k} \lambda_j \int g_j(x) \ln g_j(x)\, dx$$

$$= K - \left(H(\bar{g}(x)) - \sum_{j=1}^{k} \lambda_j H(g_j(x)) \right) \tag{46}$$

$$= K - J(\lambda_1, \lambda_2, \ldots, \lambda_k, g_1(x), g_2(x), \ldots, g_k(x)) \tag{47}$$

which is a circle of the second type with centre at the probability distribution with density function $\bar{g}(x)$ and with radius less than K by an amount equal to Jensen's difference .

These results are similar to these for the discrete-variate distributions ,

6. USE OF OTHER MEASURES OF DIRECTED DIVERGENCE

(a) If we use Havrda and Charvat's [2] measure of directed divergence, the weighted ellipse of the first type is

$$\frac{1}{\alpha-1} \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{n} p_i^{\alpha} \, q_{ij}^{1-\alpha} - 1 \right) = K \, , \; \alpha \neq 1 \, , \; \alpha > 0 \qquad (48)$$

or

$$\sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{k} \lambda_j \, q_{ij}^{1-\alpha} = K \, (\alpha-1) + 1 \qquad (49)$$

or

$$\sum_{i=1}^{n} p_i^{\alpha} \, q_i^{1-\alpha} = K (\alpha - 1) + 1 \, ; \quad \sum_{j=1}^{k} \lambda_j \, q_{ij}^{1-\alpha} = q_i^{1-\alpha} \qquad (50)$$

or

$$\sum_{i=1}^{n} p_i^{\alpha} \, \overset{\approx}{q}_i^{\,1-\alpha} = A^{\alpha-1} \, (K (\alpha-1) + 1) ; \; \overset{\approx}{q}_i = \frac{q_i}{A} \, ; \; A = \sum_{i=1}^{n} q_i \qquad (51)$$

or

$$\frac{1}{\alpha-1} \left(\sum_{i=1}^{n} p_i^{\alpha} \, \overset{\approx}{q}_i^{\,1-\alpha} - 1 \right) = \frac{A^{\alpha-1} (K (\alpha-1) + 1) - 1}{\alpha-1} \qquad (52)$$

which is again a circle of the first type with centre at $\overset{\approx}{Q}$ and radius

$[A^{\alpha-1} (K (\alpha - 1) + 1) - 1] / (\alpha-1)$ or $A^{\alpha-1} \left(k + \dfrac{1-A^{1-\alpha}}{\alpha-1} \right)$.

(b) In this case, the weighted ellipse of the second type is

$$\frac{1}{\alpha-1} \sum_{j=1}^{k} \lambda_j \left(\sum_{i=1}^{n} q_{ij}^{\alpha} \, p_i^{1-\alpha} - 1 \right) = K, \; \alpha \neq 1, \; \alpha > 1 \qquad (53)$$

or $$\sum_{i=1}^{n} p_i^{1-\alpha} \sum_{j=1}^{k} \lambda_j \, q_{ij}^{\alpha} = K(\alpha - 1) + 1 \tag{54}$$

or $$\sum_{i=1}^{n} p_i^{1-\alpha} \, \bar{q}_i^{\alpha} = K(\alpha-1) + 1 \;;\; \bar{q}_i^{\alpha} = \sum_{j=1}^{k} \lambda_j \, q_{ij}^{\alpha} \tag{55}$$

or $$\frac{1}{\alpha-1}\left(\sum_{i=1}^{n} p_i^{1-\alpha} \, \bar{\bar{q}}_i^{\alpha} - 1 \right) = \frac{B^{\alpha}\,(K(\alpha-1)+1)-1}{\alpha-1} \,,$$

$$\bar{\bar{q}}_i = \frac{\bar{q}_i}{\sum_{i=1}^{n} \bar{q}_i} \,,\; B = \sum_{i=1}^{n} \bar{q}_i \tag{56}$$

so that the weighted ellipse of the second type is a circle of the second type with centre at $\bar{\bar{Q}}$ and radius $(B^{\alpha} (K(\alpha-1) + 1) - 1)/(\alpha-1)$.

(c) By proceeding as in section 5, we can easily shown that for the continuous-variate distributions also, the weighted ellipses of the first and second types are circles of the first and second type, even when Havrda and Charvat's measure of directed divergence is used ,

(d) By maning $\alpha \to 1$, we can deduce results of sections 2 and 3 as limiting cases of results proved on subsections (a) and (b) above ,

(e) If we use Renyi's [9] measure of directed divergence, the weighted ellipse of the first type is

$$\frac{1}{\alpha-1} \sum_{j=1}^{k} \lambda_j \ln\left(\sum_{i=1}^{n} p_i^{\alpha} q_{ij}^{1-\alpha} \right) = K \,,\; \alpha \neq 1,\; \alpha \geqslant 0 \tag{57}$$

or $$\ln \prod_{j=1}^{k} \left(\sum_{i=1}^{n} p_i^{\alpha} q_{ij}^{1-\alpha} \right)^{\lambda_j} = K(\alpha-1) \tag{58}$$

weich cannot be written normally in the form

$$ln \sum_{i=1}^{n} p_i^{\alpha} \, q_i^{*1-\alpha} = K' \tag{59}$$

Thus, if we use Renyi's measure of directed divergence, the weighted ellipse of the first type is not a circle of ths first type .

(f) Similarly, if we use Kapur's ([3], [4]) or Sharma and Mittal's [11] measures of directed divergence, the weighted ellipses of the first and second type will not in general be circles of the first and second type .

(g) If, for two measures of directed divergence $D_1 (P : Q)$ and $D_2 (P : Q)$, we can write

$$\sum_{j=1}^{k} \lambda_j D_1 (P : Q_j) = A_1 D_1 (P . \bar{Q}) + B_1 , \quad \sum_{j=1}^{k} \lambda_j D_2 (P : Q_j)$$

$$= A_2 D (P : \bar{\bar{Q}}) + B_2 \tag{60}$$

where $\bar{Q}$, $\bar{\bar{Q}}$ are proper probability distributions and $A_1 , A_2 > 0$, then if

$$D (P : Q) = a_1 D_1 (P : Q) + a_2 D_2 (P : Q) , \; a_1 > 0, \; a_2 > 0 \tag{61}$$

we get

$$\sum_{j=1}^{k} \lambda_j D (P : Q_j) = a_1 A_1 D_1 (P : \bar{Q}) + a_2 A_2 D_2 (P : \bar{\bar{Q}})$$
$$+ a_1 B_1 + a_2 B_2 \tag{62}$$

and the *RHS* of (62) will not in general be of the form $B D(P : \bar{\bar{Q}}) + C$ and as such even if for two measures of directed divergence , $D_1(P : Q)$ and $D_2(P : Q)$, a weighted ellipse becomes a circle, this will not in general happen for a directed divergence measure which is a convex linear combination of $D_1 (P :Q)$ and $D_2 (P : Q)$.

In particular for Sharma and Taneja's [12] measure which is a convex linear combination of Havrda and Charvat's measures, whenever it is valid, a weighted ellipse need not be a circle .

7. CONCLUDING REMARKS

We have proved that when we use Havrda and Charvat's measure of directed divergence or its limiting case viz . Kullback and Leibler's measure of directed divergence, as a measure of distance, then every weighted ellipse whether of first or second type, is a circle of the same type . However this result does not hold for most other measures of directed divergence .

The result will hold for the measure

$$D(P:Q) = \sum_{i=1}^{n} \phi(p_i, q_i) \tag{63}$$

if
$$\sum_{j=1}^{k} \lambda_j \phi(p, q_j) = a\,\phi(p, \bar{q}) + b \tag{64}$$

where $\bar{q}$ is some weighted mean of $q_1, q_2, \ldots, q_k$.

Our other main results concern the minimum and maximum possible values of

$\sum_{j=1}^{k} \lambda_j D(P:Q_j)$ and $\sum_{j=1}^{k} \lambda_j D(Q_j:P)$ for variations in P .

(i) For Kullback-Leibler [8] measure $D(P:Q)$

$$\min_P \left[\sum_{j=1}^{k} \lambda_j D(P:Q_j) \right] = -\ln \sum_{i=1}^{n} q_i , \quad \max_P \left[\sum_{j=1}^{k} \lambda_j D(P:Q_j) \right]$$

$$= -\ln \min(q_1, q_2, \ldots, q_n) \tag{65}$$

$$\min_{P}\left[\sum_{j=1}^{k} \lambda_j D(Q_j : P)\right] = J(\lambda_1, \lambda_2, \ldots, \lambda_k ; Q_1, Q_2, \ldots, Q_k),$$

$$\max_{P}\left[\sum_{j=1}^{k} \lambda_j D(Q_j : P)\right] = \infty \tag{66}$$

$$\min_{f}\left[\sum_{j=1}^{k} \lambda_j D(f : g_j)\right] = -\ln\int g(x)\,dx ; \ \max_{f}\left[\sum_{j=1}^{k} \lambda_j D(f : g_j)\right] = \infty \tag{67}$$

$$\min_{f}\left(\sum_{j=1}^{k} \lambda_j D(g_j : f)\right) = J(\lambda_1, \ldots, \lambda_k ; g_1, g_2, \ldots, g_k) ;$$

$$\max_{f}\left(\sum_{j=1}^{k} \lambda_j D(g_j : f)\right) = \infty \tag{68}$$

(ii) For Havrda and Charvat's 2-measure of directed divergence

$$\min_{P}\left[\sum_{j=1}^{k} \lambda_j D(P : Q_j)\right] = \frac{A^{1-\alpha} - 1}{\alpha - 1} =$$

$$= \frac{\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{k} \lambda_j q_{ij}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} - 1}{\alpha - 1} \tag{69}$$

$$\max_{P}\left[\sum_{j=1}^{k} \lambda_j D(P : Q_j)\right] = \max_{i} \frac{q_i^{1-\alpha} - 1}{\alpha - 1} \tag{70}$$

$$\min_{P}\left[\sum_{j=1}^{k} \lambda_j D(Q_j : P) = \frac{\left(\sum_{i=1}^{n} \bar{q}_i\right)^{-\alpha}}{\alpha - 1}\right. . \tag{71}$$

As $\alpha \to 1$, these give results for Kullback - Leibler's measure [8] .

Similarly expressions can be found for the continuous-variate case .

The RHS of (71) can be regarded as a generalisation of Jensen's difference .

The important points to note is that both for Kullback-Leibler and Havrda and Charvat's measures, both the weighted averages of directed divergences to and from a given set of probability distributions have a minimum value and this result is true for both discrete and continuous-variate distributions ,

The geometry of the probability space and some optimization problems therein have been discussed by Kapur ([5] and [6]) .

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ON THE DEGREE OF APPROXIMATION OF CONTINUOUS FUNCTIONS

By

Prem Chandra

School of Studies in Mathematics, Vikram University,

Ujjain - 456010, M. P., India

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1. INTRODUCTION

Let f be 2π–periodic and L–integrable over $[0, 2\pi]$, and let the Fourier series associated with f at a point x be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Suppose that a function f is defined on a closed interval I. If for some α, $0 < \alpha \leqslant 1$, there is a constant M such that

$$(1{\cdot}1) \qquad | f(x+h) - f(x) | \leqslant M \, | h |^{\alpha} \quad (x, x+h \in I),$$

we say that f belongs to the class Lip α on I with constant M and write

$$(1.2) \qquad f \in Lip\,(\alpha, M, I) .$$

Hence onward, I will denote the closed interval $[0, 2\pi]$.

Let $\sum_{n=0}^{\infty} c_n$ be a given series and let $s_n = c_0 + c_1 + \ldots + c_n$.

Then the (E, q) $(q > 0)$–means or $\{ s_n \}$ are defined by ([4], p.180)

$$(1.3) \qquad (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k$$

Also Borel's exponential mean of $\{ s_n \}$ is given by ([4], *p*. 182)

$$(1.4) \quad e^{-p} \sum_{n=0}^{\infty} (p_n / n\,!) \, s_n \quad (p > 0) .$$

The $(E, q) (q > 0)$ and Borel's exponential means are reqular ([4], *p*. 179 and *p*. 80) . Throughout, we write

$$(1.5) \quad \phi_x (t) \; = \tfrac{1}{2} \{ f (x+t) + f (x-t) - 2f (x) \}$$

$$(1.6) \quad s_n (x) \; = \tfrac{1}{2} a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx)$$

$$(1,7) \quad P(q, \mathrm{t}, n) = (1 + q)^{-n} (1 + q^2 + 2q \cos t)^{n/2}$$

$$(1.8) \quad Q (q, t) = \tan^{-1} \{ \sin t \, / \, (q + \cos t) \}$$

$$(1.9) \quad A = 2q (\pi (1+q))^{-2} \; .$$

We also write $t (n, q, x)$ and $T (p, x)$ for $(E, q) (q > 0)$ and Borel's exponential means of the sequence $\{ s_n (x) \}$, respectively .

It has been proved in [2] and]3] (also see [6]) that the degree of approximation of $f \in Lip (\alpha, M, I) (0 < \alpha \leqslant 1)$ by Borel's exponential and $(E, q) \; (q > 0)$ means are $O (p^{-\alpha/2}$ and $O (n^{-\alpha/2})$ respectively, none of which is of the Jackson order . However, Chiu and Holland (see Holland [5], *p*. 370) proved that, for $0 < \alpha < 1$, one can get the Jackson order under some additional conditions on f. In this paper, we assume different type of additional conditions on $f \in Lip (\alpha, M, I), 0 < \alpha \leqslant 1$, to get the Jackson order by these means. Precisely, we prove the following theorems :

THEOREM 1. *Let f be 2π-periodic and led* $f \in Lip (\alpha, M, I)$. *If*

(1.10) $t^{-1} (\phi_x (t) + Mt^{\alpha})$ *is a non-increasing function of t in* $(0,\delta)$, *where* $0 < \delta < \pi/4$, *then the degree of approximation of* f, *by* (E, q) $(q > 0)$ *means of its Fourier series, is given by*

(1.11)
$$\sup_{0 \leqslant x \leqslant 2\pi} | f(x) - t\,(n, q, x)\,| = O\,(n^{-\alpha})\,.$$

THEOREM 2. *Under the hypotheses of Theorem* 1, *the degree of approximation of* f, *by Borel*'s *exponential means of its Fourier series, is given by*

(1.12)
$$\sup_{0 \leqslant x \leqslant 2\pi} | f(x) - T\,(p, x)\,| = O\,(p^{-\alpha})\,.$$

2. LEMMA

We shall use the following lemma in the proof of Theorem 1 :

LEMMA. *Let* $0 < t \leqslant \pi$ *and* $q > 0$. *Then*

$$(1 + q)^{-n}\,(1 + q^2 + 2q \cos t\,)^{n/2} \leqslant exp\,(-Ant^2)\,.$$

For a proof, see Chandra ([3], Lemma 1) .

3. PROOF OF THEOREM 1

Proceeding as in [3], we obtain

$$\pi\,(\,t\,(\,n,q,x\,) - f\,(x)\,)$$

$$= \int_0^{\pi} \frac{\phi_x\,(t)}{\sin \frac{1}{2} t} \left\{ (\,1+q\,)^{-n} \sum_{k=0}^{n} \binom{c}{k}\, q^{n-k} \sin\,(\,k + \tfrac{1}{2}\,)\,t \right\} dt \;.$$

Now we express the integral $\int_0^{\pi}$ as a sum of sub-integrals $\int_0^{1/n}$, $\int_{1/n}^{\delta}$ and $\int_{\delta}^{\pi}$, $0 < \delta < \pi/4$, and denote these by I_1 , I_2 , and I_3 , respectively . Then

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_1 |$$

$$\leqslant \sup_{0 \leqslant x \leqslant 2\pi} \int_0^{1/n} \frac{| \phi_x (t) |}{\sin \frac{1}{2}t} \left\{ (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \right\} dt$$

$$= \sup_{0 \leqslant x \leqslant 2\pi} \int_0^{1/n} | \phi_x (t) / \sin \tfrac{1}{2}t | \, dt$$

$$= O \left\{ \int_0^{1/n} t^{\alpha-1} \, dt \right\} \quad (\text{by } (1.1))$$

$$= O (n^{-\alpha})$$

and, proceeding as in Chandra [1, (3.2)],

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_3 | \quad \sup_{0 \leqslant x \leqslant 2\pi} \int_\delta^{\pi} \frac{| \phi_x (t) |}{\sin \frac{1}{2}t} P (q, t, n)$$

$$| \sin \{ \tfrac{1}{2} t + nQ (q, t) \} | \, dt$$

$$= O \left\{ \int_\delta^{\pi} t^{\alpha-1} \exp (- Ant^2) \, dt \right\}$$

(by the lemma)

$$= O (n^{-1}) \int_\delta^{\pi} t^{\alpha-2} \left\{ - \frac{\partial}{\partial t} \exp (- Ant^2) \right\} dt$$

$$= O (n^{-1}) .$$

Next

$$| I_2 | \leqslant 2 \left| \int_{1/n}^{\delta} t^{-1} \phi_x (t) \; P (q, t, n) \; \sin \{ nQ (q, t) + \tfrac{1}{2} t \} \, dt \right|$$

$$+ \left| \int_{1/n}^{\delta} \phi_x (t) (\operatorname{cosec} \tfrac{1}{2}t - 2/t) P (q, t, n) \sin \{ \tfrac{1}{2}t + nQ (q, t) \} \, dt \right|$$

$$= 2\,|\,I_{2,1}\,| + |\,I_{2,2}\,|\,,\ \text{say}\,,$$

and

$$\sup_{0 \leqslant x \leqslant 2\pi} |\,I_{2,2}\,| = O\left\{ \int_{1/n}^{\delta} t^{1+\alpha}\ P(q, t, n)\ dt \right\}$$

$$= O(n^{-1}) \int_{1/n}^{\delta} t^{\alpha} \left\{ -\frac{\partial}{\partial t}\ exp(-Ant^2) \right\} dt \quad (\text{as for } I_3)$$

$$= O(n^{-1})\ .$$

Since

(3.1) $\quad \phi_x(t) + M\,t^{\alpha} \geqslant 0\,,$

$$|\,I_{2,1}\,| \leqslant \left| \int_{1/n}^{\delta} t^{-1} (\phi_x(t) + Mt^{\alpha})\, P(q,t,n,)\, sin\, \{nQ(q,t) + \tfrac{1}{2}t\}\ dt \right| +$$

$$+\ M \left| \int_{1/n}^{\delta} t^{\alpha-1} P(q, t, n)\ sin\ \{ nQ(q, t) + \tfrac{1}{2}t \}\ dt \right|$$

$$= |\,I_{2,1,1}\,| + M\,|\,I_{2,1,2}\,|\ ,\ \text{say}\ .$$

$P(q, t, n)$ is positive and non-increasing in $(1/n, \delta)$; therefore, by the second mean value theorem,

$$I_{2,1,2} = u^{1-\alpha}\, P(q, n^{-1}, n) \int_{1/n}^{\delta'} sin\,(nQ(q,t) + \tfrac{1}{2}t)\ dt \quad (1/n < \delta' < \delta)\ .$$

Since

$$\left\{ \frac{d}{dt}\ Q(q, t) + \frac{1}{2n} \right\}^{-1} = \left\{ \frac{q\ cos\ t + cos\ 2t}{1 + q^2 + 2q\ cos\ t} + \frac{1}{2n} \right\}^{-1}$$

is an increasing function of t ,

$$\int_{1/n}^{\delta'} sin\ \{ nQ(q, t) + \tfrac{1}{2}\, t \}\ \ dt$$

$$= \int_{1/n}^{\delta'} \left\{ n \frac{d}{dt} \; Q(q,t) + \tfrac{1}{2} \right\}^{-1} \frac{d}{dt} \left\{ - \cos (nQ(q,t) + \tfrac{1}{2}t) \right\} dt$$

$$= n^{-1} \int_{1/n}^{\delta'} \left\{ \frac{d}{dt} \; Q(q,t) + \frac{1}{2n} \right\}^{-1} \frac{d}{dt} \left\{ -\cos (nQ(q,t) + \tfrac{1}{2}t) \right\} dt$$

$$(1/n < \eta < \delta')$$

$$= O (n^{-1}) \, ,$$

we obtain

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_{2,1,2} | = O (n^{-\alpha})$$

Finally, in view of (3.1) and (1.10), and the second mean value theorem, we have

$$I_{2,1,1} = n \{ \phi_x (1/n) + M n^{-\alpha} \} \, P (q, n^{-1}, n)$$

$$\int_{1/n}^{\delta''} \sin \{ nQ (q, t) + \tfrac{1}{2}t \} \, dt \quad (1/n < \delta'' < \delta)$$

$$= O (n^{-\alpha}) \, .$$

Thus

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_{2,1,1} | = G (n^{-\alpha}) \, .$$

Collecting the results, we obtain a proof of Theorem 1 .

4. PROOF OF THEOREM 2

Proceeding as in [2] , we have

$$\pi (T (p, x) - f (x)) = \int_0^\pi \frac{\phi_x (t) \sin \{ \frac{1}{2}t + p \sin t \}}{\sin \frac{1}{2}t \exp (p (1 - \cos t))} \, dt$$

Now we express the integral $\int_0^\pi$ as a sum of sub-integrals $\int_0^{1/p}$,

$\int_{1/p}^{\delta}$ and $\int_{\delta}^{\pi}$, $0 < \delta < \pi/4$, and denote these by J_1, J_2 and J_3, respectively . Then proeeeding as in [2] . we obtain that

$$\sup \{ \mid J_1(x) \mid : 0 \leqslant x \leqslant 2\pi \} = O(p^{-\alpha})$$

and

$$\sup \{ \mid J_3(x) \mid : 0 \leqslant x \leqslant 2\pi \}$$

$$\leqslant \sup_{0 \leqslant x \leqslant 2\pi} \int_{\delta}^{\pi} \frac{\mid \phi_x(t) \mid}{2p \sin t \sin \frac{1}{2}t} \left\{ - \frac{\partial}{\partial t} \left\{ (-2p \sin^2 \tfrac{1}{2}t) \right\} dt \right.$$

$$= O(p^{-1}) \int_{\delta}^{\pi} t^{\alpha-2} \left\{ - \frac{\partial}{\partial t} \exp(-2p \sin^2 \tfrac{1}{2}t) \right\} dt$$

$$= O(p^{-1}) .$$

And

$$\mid J_2 \mid \leqslant 2 \left| \int_{1/p}^{\delta} t^{-1} \phi_x(t) \frac{\sin(\frac{1}{2}t + p \sin t)}{\exp(p(1 - \cos t))} dt \right|$$

$$+ \left| \int_{1/p}^{\delta} \phi_x(t) (\operatorname{cosec} \tfrac{1}{2}t - 2/t) \frac{\sin(\frac{1}{2}t + p \sin t)}{\exp(p(1 - \cos t))} dt \right|$$

$$= 2 \mid J_{2,1} \mid + \mid J_{2,2} \mid, \text{ say} ,$$

where

$$\sup_{0 \leqslant x \leqslant 2\pi} \mid J_{2,2} \mid = O \left\{ \int_{1/p}^{\delta} t^{1+\alpha} \exp(-2p \sin^2 \tfrac{1}{2}t) dt \right\}$$

$$= O(p^{-1}) \int_{1/p}^{\delta} t^{\alpha} \left\{ - \frac{\partial}{\partial t} \exp(-2p \sin^2 \tfrac{1}{2}t) dt \right\}$$

$$= O(p^{-1})$$

and, using the arguments used for estimating

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_{2,1} |$$

of Theorem 1, we easily obtain

$$\sup_{0 \leqslant x \leqslant 2\pi} | J_{2,1} | = O (p^{-\alpha}) .$$

This proves Theorem 2 completely .

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STABILITY OF STRATIFIED FLUID IN POROUS MEDIUM IN THE PRESENCE OF VARIABLE MAGNETIC FIELD AND UNIFORM ROTATION

By

R. C. SHARMA
Department of Mathematics, Himachal Pradesh University,
Shimla - 171 005, Himachal Pradesh, India

and

J. N. MISRA
Department of Mathematics, Government College,
Bilaspur - 174 001, Himachal Pradesh, India

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ABSTRACT

A study has been made of the stability of stratified fluid in porous medium in the presence of variable horizontal magnetic field and uniform rotation. Both the density and the magnetic field are assumed to be exponentially varying. For stable stratification, the system is found to be stable for disturbances of all wave-numbers. The magnetic field succeeds in stabilizing potentially unstable stratification for small wave-length perturbations $k > (g\beta)^{1/2} \sec\theta / V$ which were unstable in its absence. The long wave-length perturbations satisfying $k^2 < \frac{g\beta \sec^2\theta}{V^2} - \frac{m^2\pi^2}{d^2}$ remain unstable and are not stabilized by magnetic field.

1. INTRODUCTION

Chandrasekhar [1] has given a detailed account of the Rayleigh-Taylor instability under varying assumptions of hydrodynamics and hydromagnetics . The magnetic field has been considered to be constant and uni-directional throughout . Gupta [2] has studied the stability of a horizontal layer of a perfectly conducting fluid, with continuous density and viscosity stratification in the presence of a horizontal magnetic field (constant as well as variable) . When a fluid slowly permeates a porous material, the gross effect is represented by the usual Darcy's law . As a result of this, the usual viscous term in the equations of fluid motion is replaced by the resistance term (μ/k_1) v, where μ is the viscosity of the fluid, k_1 the permeability of the medium and v the velocity of the fluid, calculated from Darcy's law . The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [5] . Sharma [4] has studied the effect of uniform magnetic field and uniform rotation on the stability of two superposed fluids in porous medium .

Usually the magnetic field has a stabilizing effect on the instability . However, Kent [3] has studied the effect of a horizontal magnetic field which varies in the vertical direction on the stability of parallel flows and has shown that system is unstable under certain conditions, while in the absence of magnetic field the system is known to be stable . In the interior of the Earth, the magnetic field may be variable and may altogether alter the nature of the instability . The Coriolis force also plays an important role on the stability of geophysical phenomena .

The present paper, therefore , deals with the stability of stratified fluid in porous medium in the presence of variable magnetic field and uniform rotation .

2. Perturbation Equations

Here we study the Rayleigh-Taylor instability of a fluid with continuous density stratification in porous medium in the presence of rotation and a variable horizontal magnetic field. The fluid is considered to be heterogeneous, incompressible, infinitely conducting and infinitely extending so that the free surface is almost horizontal. The fluid is acted on by gravity g $(0,0,-g)$, a uniform rotation $\vec{\Omega}(0,0,\Omega)$ and a variable horizontal magnetic field $\mathbf{H}(H_0(z), 0,0)$.

Let ρ, p and $\mathbf{v}(u, v, w)$ denote respectively the density, pressure and velocity of the fluid; μ_e is the magnetic permeability and ϵ is the medium porosity. Then the hydromagnetic equations relevant to the problem are

$$\frac{\rho}{\epsilon}\left[\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\epsilon}(\mathbf{v}.\nabla)\mathbf{v}\right] = -\nabla p + \frac{\mu_e}{4\pi}(\nabla \times H) \times H + \rho g$$

$$- \frac{\mu}{k_1}\mathbf{v} + \frac{2\rho}{\epsilon}(\mathbf{v} \times \vec{\Omega}), \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2}$$

$$\epsilon \frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}), \tag{3}$$

$$\nabla . \mathbf{H} = 0. \tag{4}$$

Since the density of a particle moving with the fluid remains unchanged, we have

$$\epsilon \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho = 0, \tag{5}$$

Let $\delta\rho$, δp, $\mathbf{v}(u, v, w)$ and $\mathbf{h}(h_x, h_y, h_z)$ denote the perturbations in density ρ, pressure p, velocity $(0, 0, 0)$ and magnetic field $\mathbf{H}(H_0(z), 0,0)$ respectively. Then the linearized hydromagnetic perturbation equations are

$$\frac{\rho}{\epsilon}\frac{\partial \mathbf{v}}{\partial t} = -\nabla \delta p + \mathbf{g}\,\delta\rho - \frac{\mu}{k_1}\mathbf{v} + \frac{2\rho}{\epsilon}(\mathbf{v} \times \vec{\Omega})$$

$$+ \frac{\mu_e}{4\pi} \left[(\nabla \times \mathbf{h}) \times \mathbf{H} + (\nabla \times \mathbf{H}) \times \mathbf{h} \right] , \tag{6}$$

$$\nabla . \mathbf{v} = 0 , \tag{7}$$

$$\epsilon \frac{\partial \mathbf{h}}{\partial t} \ \nabla \times (\mathbf{v} \times \mathbf{H}) , \tag{8}$$

$$\nabla . \mathbf{h} = 0 , \tag{9}$$

$$\epsilon \frac{\partial}{\partial t} \ \delta\rho = - \omega \frac{d\rho}{dz} \ . \tag{10}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x,y and t is given by

$$exp \ (ik_x x + ik_y \mathrm{y} + nt) , \tag{11}$$

where k_x , k_y are the horizontal wave numbers, $k = (k_x^2 + \mathrm{k}_y^2)^{1/2}$ and n is the growth rate .

Equations (6)-(10), using expression (11), become

$$\frac{\rho}{\epsilon} \ n u = - \mathrm{i}k_x \ \delta p + \frac{\mu_e}{4\pi} \ h_z \, DH_0 - \frac{\mu}{k_1} \ u + \frac{2\rho\Omega}{\epsilon} \ v , \tag{12}$$

$$\frac{\rho}{\epsilon} \ n \ v = - i \ k_y \ \delta p + \frac{\mu_e H_0}{4\pi} \ (ik_x h_y - ik_y h_x) - \frac{\mu}{k_1} \ \mathrm{v} - \frac{2\rho\Omega}{\epsilon} \ u , \tag{13}$$

$$\frac{\rho}{\epsilon} \ n\omega = - D \ pd + \frac{\mu_e H_0}{4\pi} \ (ik_x h_z' - Dh_x - h_x \frac{DH_0}{H_0} - \frac{\mu}{k_1} \omega + \frac{g}{n\epsilon} (D\rho) \ \omega , \tag{14}$$

$$ik_x u + ik_y v + Dw = 0 , \tag{15}$$

$$ik_x h_x + ik_y h_y + Dh_z = 0 , \tag{16}$$

$$n\epsilon \mathbf{h} = ik_x H_0 \ \mathrm{v} - \omega \ (DH_0) \ \mathbf{I}_x , \tag{17}$$

where $\mathbf{I}_x\,(1,0,0)$ is the unit vector in the direction of x–axis and $D = d/dz$.

Multiplying eqs. (12) and (13) by $-ik_x$, $-ik_y$ respectively , adding and substituting for h_x, h_y, h_z ; we get

$$\frac{\rho}{\epsilon}\, n\, D\omega = -\,k^2\,\delta p - \frac{\mu}{k_1}\, D\omega - \frac{2\rho\Omega}{\epsilon}\,\zeta + \frac{k_x k_y}{\epsilon}\,\frac{\mu_e H_0^2}{4\pi n}\,\zeta$$

$$+\,\frac{\mu_e H_0 k^2}{4\pi n\epsilon}\,\omega\, DH_0\ , \tag{18}$$

where $\zeta\,\left(=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)$, the z-component of vorticity, is given by

$$\zeta = \frac{2\Omega D\omega}{n + \dfrac{\nu\epsilon}{k_1} + \dfrac{k_x^2 V^2}{n}}\ , \tag{19}$$

which is obtained by multiplying eqs . (12) and (13) by $-ik_y$ and $+ik_x$ respectively , adding and substituting for h_x, h_y, h_z .

$V^2 = \dfrac{\mu_e H^2}{4\pi\rho}$ is the square of the Alfvén velocity .

Eliminating δp between eqs , (14) and (18), using (19) and the relation

$$ik^2\, u = -\,(\,k_x D\omega + k_y\zeta\,) = -\left(k_x + \frac{2k_y\Omega n}{n^2 + \dfrac{\nu\epsilon}{k_1}\, n + k_x^2\, V^2}\right) D\omega\ , \tag{20}$$

we get after simplification

$$\left[1 + \frac{4\Omega^2}{n^2 + \dfrac{\nu\epsilon\, n}{k_1} + k_x^2 V^2}\right] D\,(\,\rho Dw\,) - k^2\rho\omega + \frac{\mu\epsilon}{k_1 n}\,(\,D^2 - k^2\,)\,\omega$$

$$+\frac{\mu_e k_x^2}{4\pi n^2}\left[\, D(\, H_0^2 D\omega\,) - k^2 H_0^2 \omega \,\right] = -\frac{gk^2}{n^2}\,(\, D\rho\,)\,\omega\ . \tag{21}$$

Equation (21) is the general equation formulating the effect of uniform rotaiton and a variable horizontal magnetic field on the stability of stratified fluid in a porous medium . For the case of a uniform horizontal magnetic field, eq. (21) reduces to the result (Sharma [4]) .

3. The Case of Exponentially Varying Density and Magnetic Field

Let us assume the stratifications in density and magnetic field of the form

$$\rho = \rho_1\, e^{\beta z}\ ,\quad H_0^2 = H_1^2\ e^{\beta z}\ , \tag{22}$$

where ρ_1, H_1 and β are constants and so

$$V^2 = \frac{\mu_e H_0^2}{4\pi\rho} = \frac{\mu_e H_1^2}{4\pi\rho_1}\ . \tag{23}$$

Using the stratification of the form (22), eq. (21) transforms to

$$\left[\, 1 + \frac{k_x^2 V^2}{n^2} + \frac{\nu\epsilon}{k_1 n} + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon}{k_1}\, n + k_x^2\ V^2} \right] D^2\omega$$

$$+ \left[\, 1 + \frac{k_x^2 V^2}{n^2} + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon}{k_1}\, n + k_x^2\ V^2} \right] \beta D\omega$$

$$- \left[\, 1 + \frac{k_x^2 V^2}{n^2} + \frac{\nu\epsilon}{k_1 n} - \frac{g\ \beta}{n^2} \right] k^2\omega = 0\ . \tag{24}$$

The general solution of eq. (24) is

$$\omega = A_1 \, e^{q_1 z} + A_2 \, e^{q_2 z} \; , \tag{25}$$

where A_1 , A_2 are two arbitrary constants and q_1 , q_2 are the roots of the equation

$$\left[1 + \frac{k_x^2 V^2}{n^2} + \frac{\nu\epsilon}{k_1 n} + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon}{k_1} n + k_x^2 V^2}\right] q^2$$

$$+ \left[1 + \frac{k_x^2 V^2}{n^2} + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon}{k_1} n + k_x^2 V^2}\right] \beta q$$

$$- \left[1 + \frac{k_x^2 V^2}{n^2} + \frac{\nu\epsilon}{k_1 n} - \frac{g\beta}{n^2}\right] k^2 = 0 \, . \tag{26}$$

If the fluid is supposed to be confined between two rigid planes at $z = 0$ and $z = d$, then the vanishing of w at $z = 0$ is satisfied by the choice

$$\omega = A \left(e^{q_1 z} - e^{q_2 z} \right) , \tag{27}$$

while the vanishing of w at $z = d$ reqires

$$exp \, (q_1 - q_2) d = 1 \, , \tag{28}$$

which implies that

$$(q_1 - q_2) d = 2im\pi \, , \tag{29}$$

where m is an integer .

Equation (26) gives

$$
\begin{aligned}
q_{1,2} = \Bigg[& -\frac{\beta}{2}\left(1+\frac{k_x^2V^2}{n^2}+\frac{4\Omega^2}{n^2+\frac{\nu\epsilon}{k_1}n+k_x^2V^2}\right) \\
& \pm \tfrac{1}{2}\Bigg\{\beta^2\left(1+\frac{k_x^2V^2}{n^2}+\frac{4\Omega^2}{n^2+\frac{\nu\epsilon}{k_1}n+k_x^2V^2}\right)^2 \\
& +4\,k^2\left(1+\frac{k_x^2V^2}{n^2}+\frac{\nu\epsilon}{k_1n}+\frac{4\Omega^2}{n^2+\frac{\nu\epsilon}{k_1}n+k_x^2V^2}\right)\times \\
& \times\left(1+\frac{k_x^2V^2}{n^2}+\frac{\nu\epsilon}{k_1n}-\frac{g\beta}{n^2}\right)\Bigg\}^{1/2}\Bigg]. \\
& .\left(1+\frac{k_x^2V^2}{n^2}+\frac{\nu\epsilon}{k_1n}+\frac{4\Omega^2}{n^2+\frac{\nu\epsilon}{k_1}n+k_x^2V^2}\right)^{-1}. \qquad (30)
\end{aligned}
$$

Inserting the values of q_1, q_2 from (30) in (29) and simplifying, we obtain

$$A_8n^8+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0 = 0 \,, \qquad (31)$$

Equation (31) is the dispersion relation studying the effect of rotation and the variable (exponential) horizontal magnetic field on the stability of stratified (exponentially varying density) fluid in porous medium .

If $\beta < 0$ (stable stratification) , eq . (31) does not admit of any positive root of n and the system is always stable for disturbances of all wave unmbers .

If $\beta > 0$ (unstable stratification) and if

$$V^2 < \frac{g\beta k^2}{\left(k^2+\frac{m^2\pi^2}{d^2}\right)k_x^2} , \tag{32}$$

the constant term in eq . (31) is negative , Equation (31) therefore has at least one positive real root and so the system is unstable for all wave-numbers satisfying the inequality

$$k^2 < \frac{g\beta \sec^2\theta}{V^2} - \frac{m^2\pi^2}{d^2} , \tag{33}$$

where θ is the angle between k_x and k (*i.e.* $k_x = k \cos\theta$) .

If $\beta > 0$ (unstable stratification) but $k_x^2 V^2 > g\beta$, eq . (31) admits of no positive root and so the system is stable .

Thus for unstable stratification but magnetic field such that $V^2 < g\beta \sec^2\theta / (k^2 + \frac{m^2\pi^2}{d^2})$, the system is unstable for all wave-numbers satisfying $k^2 < \frac{g\beta \sec^2\theta}{V^2} - \frac{m^2\pi^2}{d^2}$.

4. RESULTS

The results of the paper are presented here .

(i) The dispersion relation studying the effect of rotation and the variable (exponential) horizontal magnetic field on the stability of stratified (exponentially varying density) fluid in porous medium has been obtained .

(ii) For stable stratification, the system is fonud to be stable for disturbances of all wave-numbers .

(iii) The magnetic field stabilizes potentially unstable stratification for small wave-length perturbations $k > (g\beta)^{1/2} \sec\theta / V$ which were unstable in its absence.

(iv) The long wave-length perturbations satisfying

$k^2 < \frac{g\beta \sec^2\theta}{V^2} - \frac{m^2\pi^2}{d^2}$ remain unstable and are not stabilized by magnetic field.

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APPENDIX

The coefficients $A_0 - A_8$ in eq. (31) are given below .

$$A_8 = \beta^2 + 4k^2 + \frac{4m^2\pi^2}{d^2} \ ,$$

$$A_7 = a \left(2\beta^2 + 16k^2 + \frac{16m^2\pi^2}{d^2} \right),$$

$$A_6 = \beta^2 (4k_x V^2 + a^2) + \frac{16m^2\pi^2}{d^2} (k_x^2 \ V^2 + a^2)$$

$$+ 4k^2 \{ 4\Omega^2 + 6a^2 + (4k_x^2 \ V^2 - g\beta) \} \ ,$$

$$A_5 = 2a\beta^2 (3k_x^2 \ V^2 + 4\Omega^2) + \frac{16m^2\pi^2}{d^2} a \ (3k_x^2 \ V^2 + 4\Omega^2 + a^2)$$

$$+ 4ak^2 \ \{ 9k_x^2 \ V^2 + 8\Omega^2 + 4a^2 + 3 (k_x^2 \ V^2 - g\beta) \} \ ,$$

$$A_4 = 2\beta^2 (3k_x^4 \ V^4 + k_x^2 \ V^2 a^2 + 8\Omega^4 + 4\Omega^2) + 4k^2 \{ 3k_x^2 \ V^2 (2k_x^2 \ V^2 - g\beta)$$

$$+ a^4 + 3a^2 (4k_x^2 \ V^2 - g\beta) + 4\Omega^2 (2k_x^2 \ V^2 - g\beta] + 4\Omega^2 a^2 \}$$

$$+ \frac{4m^2\pi^2}{d^2} \{ 2k_x^2 \ V^2 (3k_x^2 \ V^2 + 4a^2) + (4\Omega^2 + a^2)^2 + 2\Omega^2 + 2a^2 \} \ ,$$

$$A_3 = 2a\beta^2 \ k_x^2 \ V^2 (3k_x^2 \ V^2 + 4\Omega^2) + 4k^2 \ \{ 6ak_x^2 \ V^2 (2k_x^2 \ V^2 - g\beta)$$

$$+ 8a\Omega^2 k_x^2 \ V^2 + a_3 (4k_x^2 \ V^2 - g\beta) \} + \frac{16m^2\pi^2}{d^2} ak_x^2 \ V^2 (3k_x^2 \ V^2 + 4\Omega^2 + a^2),$$

$$A_2 = \beta^2 k_x^2 V^2 \, (4k_x^4 V^4 + a k_x^2 V^2 + 8\Omega^2)$$

$$+ \frac{8m^2\pi^2}{d^2} \{ 2k_x^6 V^6 + 2a^2 k_x^4 V^4 + k_x^2 V^2 (4\Omega^2 + a^2) \}$$

$$+ 4k^2 \{ k_x^4 V^4 (k_x^2 V^2 + 3a^2) + k_x^2 V^2 (k_x^2 V^2 - g\beta) (3k_x^2 V^2 + 3a^2 + 4\Omega^2) \} ,$$

$$A_1 = 2a k_x^6 V^6 (\beta^2 + 2k^2) + 12a k_x^4 V^4 (k_x^2 V^2 - g\beta) + \frac{16 a k^6 V^6 m^2 \pi^2}{d^2} ,$$

$$A_0 = 4k_x^6 V^6 \, \{ k^2 (k_x^2 V^2 - g\beta) + \frac{m^2\pi^2}{d^2} k_x^2 V^2 ,$$

and

$$a = \nu\epsilon / k_1 .$$

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COMMENTS ON ISTRATESCU'S CONVEX CONTRACTION MAPS AND MAPS WITH CONVEX DIMINISHING DIAMETERS

By

SEHIE PARK*
Department of Mathematics, Seoul National University
Seoul 151, Korea

and

B. E. RHOADES
Department of Mathematics, Indiana University,
Bloomington, Indiana 47405, U. S. A.

(*Received* : *March 6, 1987*)

In [3] and [4], V. I. Istratescu considered several classes of maps related to contraction maps by introducing convexity conditions with respect to the iterates of maps, and obtained a number or fixed point theorems on such maps. However, we notice that some of his results are closely related to our previous work ([5], [6]). In fact, in the present paper, we show that some of Istratescu's key results are obvious consequences of our previous results in [5] and [6] . Further, we obtain generalizations of some of his results .

Let f be a selfmap of a metric space (X, d) . Let $0(x)$ be the orbit of $x \in X$ under f. A point $x \in X$ is said to be regular if $0(x)$ has finite diameter . we denote

$\delta(x,y) = diam\ \{0(x) \cup 0(y)\}$ *if* x and y are regular .

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Theorem 1 ([6], Corollary to Theorem 1) . *Let f be a continuous selfmap of a complete metric space X . Suppose that*

(i) *there exists a regular point $x \in X$, and*

(ii) *for fixed integers p and q, given $\epsilon > 0$ there exist an ϵ_0, $0 < \epsilon_0 < \epsilon$, and $\delta_0 > 0$ such that for each $x, y \in X$.*

$\epsilon \leqslant \delta (x,y) < \epsilon + \delta_0$ *implies* $d (f^p x, f^q y) \leqslant \epsilon_0$.

Then f has unique fixed potnt $z \in X$ and $f^n \to z$ for each regular $x \epsilon X$.

We now show that Theorem 1 includes Theorems 1.2, 1.5, 2.3 and 2.4 of Istratescu [3] :

Let f be a continuous selfmap of a complete metric space X satisfying one of the following conditions :

(1) there exists $a_0, a_1, \ldots, a_{n-1} \in (0,1)$,

$a_0 + \ldots + a_{n-1} < 1$, such that for all $x, y \in X$,

$d (f^n x, f^n y) \leqslant a_0 d (x,y) + a_1 d (fx, fy) + \cdot \cdot + a_{n-1} d (f^{n-1} x, f^{n-1} y)$

(2) there exist $c_0 , c_1 , a_1 , a_2 , b_1, b_2 \in (0,1)$,

$c_0 + c_1 + a_1 + a_2 + b_1 + b_2 < 1$, such that for all $x, y \in X$,

$$d (f^2 x, f^2 y) \leqslant c_0 d (x,y) + c_1 d (fx, fy) + a_1 d (x, fx)$$

$$+ a_2 d (fx, f^2 x) + b_1 d (y, fy) + b_2 d (fy, f^2 y) .$$

Note that each of (1) and (2) (with $n=2$) implies

$$d (f^n x, f^n y) \leqslant c \; diam \{ x, fx , \ldots , f^n x, y, fy , \ldots, f^n y \}$$

where $c \in (0,1)$ and whence the orbit $0(x)$ of any $x \in X$ is bouuded (e. g. see B. Fisher [2]) . Therefore, we have

$$d (f^n x, f^n y) \leqslant c \delta (x,y) ,$$

and by the argument in [6], all the hypotheses of Theorem 1 are satisfied by putting $p = q = n$. Hence f has a unique fixed point $z \in X$ and $f^n x \to z$ for all $x \in X$.

Since obvious modifications of Theorem 1 hold for a generalized metric space (in which the distance can take on the value $+\infty$) and a Hausdorff uniform s space, Istrateseu's Theorems 4.2 [3] and 4.5 [4] can also be extended.

Theorem 2 [6]. *Let f be a continuous selfmap of a metric space X. Suppose that for fixed integers p and q,*

$$d(f^p x, f^q y) < \delta(x, y)$$

for each $x, y \in X$ $x \neq y$. Then f has a unique fixed point. Furthermore, for any $\alpha \in (0,1)$ there exists a metric ρ on X, relative to which f is a Banach contraction with Lipschitz constant α.

We now show that Theorem 2 includes Theorem 1.7 of Istratescu [3]:

Let f be a continuous selfmap of a compact metric space X such that there exist $a_0, a_1 \in (0,1)$, $a_0 + a_1 = 1$, and for all $x, y \in X$, $x \neq y$,

$$d(f^2 x, f^2 y) < a_0 d(x, y) + a_1 d(fx, fy).$$

Then the hypotheses of Theorem 2 are clearly satisfied.

Theorem 3. *Let f be a continuous selfmap of a metric space X and $F: X \to R$ a continuous function satisfying:*

(i) $F(fx) \leqslant F(x)$ *for all* $x \in X$,

(ii) *if* $x \neq fx$, *then* $F(fx) < F(x)$; *and*

(iii) *for some* $x \in X$, *the sequence* $\{f^n x\}$ *has a cluster point* $z \in X$.

Then z is a fixct point of f.

Proof. By (i), the sequence $\{F(f^n x)\}$ is nonincreasing. Set $z = lim_k f^{n_k} x$. By (i), we have

$$F(z) = lim_k F[f^{n_k} x] = lim_n F(f^n x)$$

$$= lim_k F[f^{n_k+1} x] = F(fz),$$

contradicting (ii) unless $z = fz$.

In the case where $F(x) = d(x, fx)$, Theorem 1 extends a well-known result of Edelstein.

We show that Theorem 1.8 of Istratescu [3] follows from Theorem 3:

Let $f : X \to X$ satisfying

(3) $d(f^2x, f^2y) \leqslant ad(fx, fy) + bd(x, y)$, $a, b, \in (0,1)$, $a+b<1$,

for all $x, y \varepsilon X$. Since $a < 1 - b$, we have

$$d(f^2 x, f^3 x) \leqslant bd(x, fx) + (1-b) d(fx, f^2 x).$$

By putting

$$F(x) = bd(x, fx) + d(fx, f^2x),$$

we have $F(fx) \leqslant F(x)$. Suppose $F(fx) = F(x)$. Then

$$bd(fx, f^2x) + d(f^2x, f^3x) = bd(x, fx) + d(fx, f^2 x),$$

and

(4) $d(f^2 x, f^3x) = bd(x, fx) + (1-b) d(fx, f^2x)$.

If $fx \neq f^2x$, then we have $d(f^2x, f^3x) > bd(x, fx) + ad(fx, f^2x)$, contradicting (3). Therefore $fx = f^2x$, which implies $f^2x = f^3x$ and, from (4) since $b > 0$, $x = fx$. Therefore (ii) is satisfied. The uniqueness of the fixed point is clear.

Corollary 1. *Let f a continuous selfmap of a metric space (X, d) and $F : X \times X \to$ R a continuous function such that*

(i) $F(fx, fy) \leqslant F(x,y)$ *for all* $x,y \in X$,

(ii } *if* $x \neq fx$, *then* $F(fx, f^2x) < F(x, fx)$, *and*

(iii) *for each* $x \in X$, *the sequence* $\{f^n x\}$ *has a cluster point* $z \in X$. *Then z is a fixed point of* f,

Proof. Replace $F(x)$ in Theorem 3 by $F(x, xf)$.

In [1], W Cheney and A. A. Goldstein obtained the above corollary for the case $F(x, y) = d(x, y)$. In this case, for each $x \in X$, $\{f^n x\}$ converges to a fixed point of f.

Theorem 1.6 of [4] follows from the corollary by defining

$$F(x, y) = (1-s)\, d(x, y) + d(fx, fy).$$

Theorem 4 ([5], proposition 1). *Let X be a complete metric space and* $\{A_n\}$ *a sequence of nonempty subsets of X such that*

$A_{n+1} \subset A_n$ *for each n. Let* $d_n = diam(A_n)$ *satisfy the following:*

For aay $\epsilon > 0$, *there exists a* $\delta > 0$ *snch that*

$$\epsilon \leqslant d_n < \epsilon + \delta \text{ impli s } d_{n+1} < \epsilon .$$

Then $A = \bigcap_n A_n$ *is a singleton.*

Corollary 2. ([3], Theorem 3.3). *Let f be a selfmap of a complete metric space X such that for any bounded subset M of X, there exist an* $a \in [0,1]$ *and an integer* $n = n(M)$ *such that*

$$diam\,(f^n(M)) \leqslant a\; diam\,(M).$$

If X is bounded, then f has a unique fixed point.

Proof. *Let* $A_1 = X$, $A_2 = f^{n(A_1)} A_1, \ldots, A_m = f^{n(A_{m-1})} A_{m-1}, \ldots,$ and $d_n = diam\,(A_n)$. Since $d_n \leqslant a d_{n-1}$, by Theorem 4, $A = \bigcap_n \bar{A}_n$ is a singleton, which is the unique fixed point of f.

Since an obvious modification of Theorem 4 holds for a Hausdorff uniform space, Istratescu's Theorem 4.8 [4] can also be extended .

Theorem 5. *Let* f *be a selfmap of a topological space* X *and* $F : X \to \mathbb{R}$ *a l. s. c. function . Suppose that*

(i) *if* $x \neq fx$, $x \in X$, *then* $F(fx) < F(x)$, *and*

(ii) *the range of* f *is relatively countably compact . Then* f *has a fixed point .*

Proof . *Let* $A := \overline{fX}$. Since A is countably compact and F is *l. s. c.* on A, F attains *its* infimum on A ; that is, there exists a $u \in A$ such that $F(u) \leqslant F(x)$ for all $x \in A$. Suppose that $u \neq fu$. Then $F(fu) < F(u)$, which is a contradiction .

To prove Theorem 4.12 of [4], set $F(x) = g(x, fx)$.

Since continuous densifying selfmaps of a complete metric space have relatively compact orbits whenever the orbits are bounded , Theorem 5 can also be applied to such maps .

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SOME INEQUALITIES FOR THE MULTIVARIABLE *H*-FUNCTION

By

C. K. SHARMA and A. C. LAMBA

Department of Mathematics, A. P. S. University,
Rewa - 486003, M. P., India

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ABSTRACT

In this paper, we establish five inequalities for the multivariable *H*-function with the help of certain integrals evaluated by Gupta and Jain [2] and some known inequalities given by Luke [3]. The inequalities which were given by Raina and Koul [4], and Anandani and Singh [1], are Particular cases of these results.

1. INTRODUCTION AND NOTATIONS

The multivariable *H*-function which was introduced by Srivastava and Panda ([7], [8]) is an extension of the multivariable *G*-function. This multivariabe *H*-function includes Fox's *H*- and Meijer's *G*-functions of one and two variables, the generalized Lauricella function of Srivastava and Daoust [5], Appell functions, the Whittaker functions, etc. Therefore, the results established in this paper are of a general character.

The multivariable *H*-function is defined by Srivastava and Panda ([7], [8]) and is represented in the manner already detailed by Srivastava, Gupta and Goyal [6, p. 251]. Throughout this paper, we also use the shorthand notations as follows :

$$(1.1)\quad \nu = \frac{\lambda}{\delta}\ ,\ \theta = \frac{\alpha^{P}}{\beta^{P}},\quad \Phi = \frac{\alpha^{P}+1}{\beta^{P}+1}\ ,\ \Psi = \prod_{j=1}^{P} \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}$$

and

$$(1.2)\quad M_1 = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} + \delta_j^{(i)}\nu) \prod_{j=1}^{n_i} \Gamma(1-c_j^{(i)} - \gamma_j^{(i)}\nu)}{\prod_{j=m_i+1}^{q_i} \Gamma(1-d_j^{(i)} - \delta_j^{(i)}\nu) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} + \gamma_j^{(i)}\nu)}$$

$$\frac{\prod_{j=1}^{n} \Gamma(1-a_j - \sum_{i=1}^{r} \alpha_j^{(i)}\nu)}{\prod_{j=n+1}^{p} \Gamma(a_i + \sum_{i=1}^{r} \alpha_j^{(i)}\nu) \prod_{j=1}^{q} \Gamma(1-b_j \sum_{j=1}^{r} \beta_j^{(i)}\nu)}$$

and M_2 represents the expression M_1, where ν has been replaced by $\nu + 1/\delta$.

${}^{*}H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$ represents the same expression as for $H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$ (*see* [6,p. 251])

only z_i, m_i, p_i q_i, $(c_j^{(i)}, \gamma_j^{(i)})_{1,p_i}$ and $(d_j^{(i)}, \delta_j^{(i)})_{1,q_i}$ are replaced by suitable parameters as needed in the next section.

2. MAIN RESULTS

The following inequalities are established for real numbers $z_1, z_2, \ldots, z_r > 0$; $\delta > 0$; $\sigma > 0$; $\beta_j \geqslant \alpha_j \geqslant 0$ ($j = 1, 2, \ldots, P$):

$$(2.1)\quad (\sigma\theta)^{-\lambda}\ {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\ q_i+1}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)}, \gamma_j^{(i)})_{1,p_i} \\ (1-\lambda,\delta),\ (d_j^{(i)}, \delta_j^{(i)})_{1,q_i} \end{matrix}\right]$$

$$< \frac{\Psi}{\Gamma(\sigma)} \; {}^{*}H^{m_i+P+1,\, n_i+1}_{p_i+P+1,\, q_i+P+1}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\,\middle|\, \begin{matrix} (1-\lambda,\delta),\ (\beta_j-\lambda,\delta)_{1,P},\ (c_j^{(i)},\gamma_j^{(i)})_{1,p_i} \\ (1-\lambda,\delta),\ (\alpha_j-\lambda,\delta)_{1,P},\ (d_j^{(i)}\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right]$$

$$< \left\{1-\frac{2\sigma\theta}{(\sigma+1)\Phi}\right\} M_1\, \delta^{-1}\, z_i^{-\nu}$$

$$+\ \sigma\theta \left\{\frac{(\sigma+1)\Phi}{2}\right\}^{(-\lambda+1)} \; {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\, q_i+1}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_j \left\{\dfrac{(\sigma+1)\Phi}{2}\right\}^{-\delta} \\ \vdots \\ z_r \end{matrix}\,\middle|\, \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ \\ (1-\lambda,\delta),\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right],$$

$$(2.2) \quad \frac{\theta^{-\lambda}}{\Gamma(\sigma)} \; {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\, q_i+1}\left[\begin{matrix} z_1 \\ \vdots \\ z_i\, \theta^{-\delta} \\ \vdots \\ z_r \end{matrix}\,\middle|\, \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (\sigma-\lambda,\delta),\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right]$$

$$< \frac{\Psi}{\Gamma(\sigma)} \; {}^{*}H^{m_i+P+1,\, n_i+1}_{p_i+P+1,\, q_i+P+1}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\,\middle|\, \begin{matrix} (1-\lambda,\delta),\ (\beta_j-\lambda,\delta)_{1,P} \\ (\sigma-\lambda,\delta),\ (\alpha_j-\lambda,\delta)_{1,P} \end{matrix}\right.$$

$$\left.\begin{matrix} (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right]$$

$$< (1-\theta)\, M_1\, \delta^{-1}\, z_i^{-\nu}$$

$$+\ \frac{\theta}{\Gamma(\sigma)} \; {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\, q_i+1}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\,\middle|\, \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (\sigma-\lambda,\delta),\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right],$$

$$(2.3)\quad M_1\,\delta^{-1}\,z_i^{-\nu}-\sigma\,\theta\left(1-\frac{\Phi}{2}\right)M_2\,\delta^{-1}\,z_i^{-[\nu+(1/\delta)]}$$

$$-\frac{\sigma\,\theta\,\Phi}{2\,\Gamma(\sigma+1)}\;{}^{*}H^{m_i+1,\,n_i+1}_{p_i+1,\,q_i+1}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(-\lambda,\delta),\,(c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ (\sigma-\lambda,\delta),\,(d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$\leq\frac{\Psi'}{\Gamma(\sigma)}\;{}^{*}H^{m_i+P+1,\,n_i+1}_{p_i+P+1,\,q_i+P+1}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(1-\lambda,\delta),\;(\beta_i-\lambda,\delta)_{1,P},\\ (\sigma-\lambda,\delta),\;(\alpha_j-\lambda,\delta)_{1,P},\end{matrix}\right.$$

$$\left.\begin{matrix}(c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$<M_1\,\delta^{-1}\,z_i^{-\nu}-\frac{\sigma\theta}{\Gamma(\sigma+1)}\left(\frac{\Phi}{2}\right)^{-(\lambda+1)}$$

$${}^{*}H^{m_i+1,\,n_i+1}_{p_i+1,\,q_i+1}\left[\begin{matrix}z_1\\ \vdots\\ z_i\left(\frac{\Phi}{2}\right)^{-\delta}\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(-\lambda,\delta),\,(c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ (\sigma-\lambda,\delta),\,(d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right],$$

$$(2.4)\quad\left(-\frac{1}{\sigma}\right)M_1\,\delta^{-1}\,z_i^{-\nu}+\left\{\left(\frac{\sigma+1}{\sigma}\right)\left(\frac{\Phi}{\sigma+1}\right)^{-\lambda}\right\}$$

$${}^{*}H^{m_i+1,\,n_i+1}_{p_i+1,\,q_i+1}\left[\begin{matrix}z_1\\ \vdots\\ z_j\left(\dfrac{\Phi}{\sigma+1}\right)^{-\delta}\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(1-\lambda,\delta).\,(c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ \\ (1-\lambda,\delta),\,(d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$<\Gamma(\sigma)\,\Psi'\;{}^{*}H^{m_i+P+1,\,n_i+1}_{p_i+P+2,\,q_i+P+1}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(1-\lambda,\delta),\,(c_j^{(i)},\gamma_j^{(1)})_{1,p_i},\\ (1-\lambda,\delta),\,(\alpha_j-\lambda,\delta)_{1,P},\end{matrix}\right.$$

$$\left.\begin{matrix}(\sigma-\lambda,\delta),\ (\beta_j-\lambda,\delta)_{1,P}\\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$< \left(1-\frac{\theta}{\Phi}\right) M_1\, \delta^{-1}\, z_1^{-\nu}$$

$$+\frac{\theta}{\Phi}\left(\frac{\Phi}{\sigma}\right)^{-\delta} {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\, q_i+1}\left[\begin{matrix} z_1\\ \vdots\\ z_i\left(\frac{\Phi}{\sigma}\right)^{-\delta}\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i}\\ (1-\lambda,\delta),\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i}\end{matrix}\right],$$

$$\text{(2.5)}\qquad \theta^{-\lambda}\ {}^{*}H^{m_i,\, n_i+1}_{p_i+1,\, q_i}\left[\begin{matrix} z_1\\ \vdots\\ z_i\,\theta^{-\sigma}\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i}\\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$< \Psi\ {}^{*}H^{m_i+P,\, n_i+1}_{p_i+P+1,\, q_i+P}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i},\ (\beta_j-\lambda,\delta)_{1,P}\\ (\alpha_j-\lambda,\delta)_{1,P},\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$< (1-\theta)\, M_1\, \delta^{-1}\, z_i^{-\nu} + \theta\ {}^{*}H^{m_i,\, n_i+1}_{p_i+1,\, q_i}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\;\middle|\;\begin{matrix}(1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i}\\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

Proofs : To prove (2.1), we replace z by $z_i t$ in (4.10) of Luke [3,p.53], multiply throughout by

$$t^{\lambda-1}\, H\left[\begin{matrix} z_1\\ \vdots\\ z_i\, t^{\delta}\\ \vdots\\ z_r\end{matrix}\right]$$

and integrate with respect to t between 0 to ∞. Now, evaluating the integrals involved on the lines of (5.1) of Gupta and Jain [2,*p*. 601] , we get the result (2.1) .

Proceeding on similar lines, the inequalities (2 2) , (2.3) , (2.4) and (2.5) can also be established, using the results of Luke [3, pp. 55 and 57, Equations (4.20) , (4.22) , (5.1) and (5.5)] .

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A RECURRENCE RELATION FOR THE *H*-FUNCTION OF SEVERAL COMPLEX VARIABLES

By

V. B. L. Chaurasia and S. C. Sharma

Department of Mathematics, University of Rajasthan,
Jaipur-302005, Rajasthan, India

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ABSTRACT

The aim of the present note is to establish a recurrence relation for the *H*-function of several complex variables due to H. M. Srivastava and R. Panda (see [3] , [4] , and [5]) .

1. INTRODUCTION AND THE MAIN RESULTS

For the *H*-function of several complex variables defined by Srivastava and Panda ([4] and [5] ; see also [3], *p.* 251), we derive the recurrence relation :

$$(1)\quad H^{0,\,\epsilon\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\,\ldots\,;\,[B^{(r)},\,D^{(r)}]}\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}$$

$$= \frac{1}{2\pi i}\left\{ e^{i\pi a_{\epsilon+1}}\, H^{0,\,\epsilon+1\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\,\ldots\,;\,[B^{(r)},\,D^{(r)}]}\begin{pmatrix} z_1\, e^{-i\pi\,\theta'_{\epsilon+1}} \\ \vdots \\ z_r\, e^{-i\pi\,\theta^{(r)}_{\epsilon+1}} \end{pmatrix}\right.$$

$$\left. - e^{-i\pi a_{\epsilon+1}}\, H^{0,\,\epsilon+1\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,[B',\,D']\,;\,\ldots\,;\,[B^{(r)},\,D^{(r)}]}\begin{pmatrix} z_1 e^{i\pi\theta'_{\epsilon+1}} \\ \vdots \\ z e^{i\pi\theta^{(r)}_{\epsilon+1}} \end{pmatrix}\right\},$$

where $i=\sqrt{-1}$, and the usual existence conditions for each H-function are assumed to be satisfied .

2. PROOF OF THE RESULT (1)

The recurrence relation (1) can be easily established by appealing to the definition ([4], *p*. 271, *Eq*. (4.1) ; [3], *p*. 251, *Eq*. (C.1) and the well-known relation (Rainville [2], *p*. 21, Theorem 8) :

$$(2) \quad \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z} = \frac{2\pi i}{e^{i\pi z} - e^{i\pi z}}$$

involving Γ-functions .

3. PARTICULAR CASES

For $r = 2$, the result (1) reduces to a recurrence relation for the H-function of two variables obtained in [1] .

Several other interesting special cases of (1) (both known and unknown) can be obtained for simpler functions of one, two and more variables .

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